# The Structure of Singular Spaces of Dimension 2

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*Abstract: In this paper, we study the structure of locally compact metric spaces of Hausdorff dimension 2. If such a space has non-positive curvautre and a local cone structure, then every simple closed curve bounds a conformal disk. On a surface (a topological manifold of dimension 2), a distance function with non-positive curvature and whose metric topology is equivalent to the surface topology gives a structure of a Riemann surface. The construction of conformal disks in these spaces uses minimal surface theory; in particular, the solution of the Plateau Problem in metric spaces of non-positive curvature.*

## 1 Introduction

One of the basic theorems of two dimensional surfaces is the existence of isothermal coordinates which implies that an oriented Riemannian surface is a Riemann surface. This is important in geometry because it gives use of powerful methods in complex analysis in the study of surfaces.

The importance of removing the regularity requirement in the theorem was realized by C.B. Morrey. Morrey weakened the regularity requirement in his Bounded Measurable Riemann Mapping Theorem:

**Theorem 1.1 (Morrey)** *Let $\Sigma$ be a surface with metric tensor given in local coordinates by bounded measurable function satisfying*

$$g_{11}g_{22} - g_{12}^2 \geq \epsilon > 0 \text{ almost everywhere.}$$

*For each $P \in \Sigma$, there is a nbhd $U$ of $P$ and a homeomorphism $h : D \to U$ satisfying the conformality relations*

$$g_{ij}\frac{\partial h^i}{\partial x}\frac{\partial h^j}{\partial x} = g_{ij}\frac{\partial h^i}{\partial y}\frac{\partial h^j}{\partial y}$$

$$g_{ij}\frac{\partial h^i}{\partial x}\frac{\partial h^j}{\partial y} = 0$$

*almost everywhere.*

A natural assumption replacing the hypothesis regarding the regularity of the metric is to require that the surface is endowed with a distance function satisfying the NPC (non-positively curved) condition. This essentially means (see section 2 for the precise definition) that geodesic triangles are *thinner* than the corresponding comparison triangles in $\mathbf{R}^2$. Furthermore, we are interested in allowing our spaces to not only be topological surfaces, but to be in a more general class of two dimensional spaces. In particular, a commonly studied class of spaces are simplicial complexes. For example, these spaces were used by Gromov and Schoen [GS] in their investigation of $p$-adic superrigidity for lattices in groups of rank one. A locally compact NPC simplicial complex has the property that for each point $P$, there is a sufficiently small $r$ (depending on $P$) so that all geodesic emanating from $P$ can be extended uniquely to length greater than $r$. If a point $P$ has this property, we will say that the cone length at $P$ is $r$, and any space with positive cone length at each point is said to have *a local cone structure*. We prove:

**Theorem 1.2** *Let $(X, d)$ be a locally compact NPC space with Hausdorff dimension 2 and a local cone structure. Let $P \in X$ and assume that there exists a simple closed curve $\tilde{\Gamma}$ which is homotopically nontrivial in $X - \{P\}$. Then there exists a conformal homeomorphism $u$ from the unit disk $D$ into $X$ such that $P \in u(D)$.*

Theorem 1.2 has the following immediate corollary:

**Corollary 1.3** *Let $X$ be a surface (i.e. a topological manifold of dimension 2) endowed with a distance function $d$ which makes $(X, d)$ into a NPC space. Furthermore, assume that the metric topology of $(X, d)$ is equivalent to the surface topology. Then for every $P \in X$, there is a neighborhood $U$ of $P$ and a conformal homeomorphism $u : D \to U$. In particular, this gives $X$ a conformal structure making it into a Riemann surface.*

Corollary 1.3 is actually implied by an old theorem of Reshetnyak's from the 1960's (see [R1]). We point out that our method of proof is quite different. In [R1], the distance function given is first approximated by distance functions induced by Riemannian metrics. The isothermal coordinates for $(X, d)$ are obtained by taking a converging sequence of local conformal maps for the approximating Riemannian metrics. Note that this sort of an argument does not generalize to spaces (such as $X$ in Theorem 1.2) which cannot be approximated by Riemannian manifolds. Our argument does not rely on this approximation technique and hence is a more general construction. We hope to find more applications of the methods outlined below in the investigation of singular spaces of non-positive curvature.

This paper is organized as follows: In Section 2, we recall the notion of curvature bounds in a metric space. Furthermore, we quote some results from the work of Korevaar-Schoen in [KS1] and of the author in [M] that is needed in this paper. Sections 3 to 6 comprise the proof of Theorem 1.2. The map $u$ in Theorem 1.2 is the solution to a Plateau Problem which we set up in Section 3. In this way, we obtain a conformal map but we still need to check that $u$ is actually a homeomorphism. Section 4 discusses the order function which is needed to construct the homogeneous approximating map of Section 5. Using the homogeneous approximating map, we prove the injectivity of $u$ in Section 6 thereby completing the proof.

*Acknowledgment:* This investigation is an extension of the author's doctoral dissertation at Stanford University. The author wishes to thank advisor Richard Schoen for his encouragement and support. She also thanks the referee for his/her careful reading and comments of the original manuscript.

# 2 Preliminaries

*Metric Spaces with Curvature Bounded from Above*

First, we review the notion of curvature bounds in a metric space $X$. We assume our metric spaces to be length spaces, i.e. for each $P$, $Q \in X$, there exists a curve $\gamma_{PQ}$ such that the length of $\gamma_{PQ}$ is exactly $d(P,Q)$. We call $\gamma_{PQ}$ a geodesic between $P$ and $Q$. We then say that $X$ is an NPC space or that $X$ has non-positive curvature in the sense of Alexandrov if geodesic triangles in $X$ are thinner than their comparison triangles in $\mathbf{R}^2$. In other words, for every $P, Q, R \in X$ and corresponding points $\bar{P}, \bar{Q}, \bar{R} \in \mathbf{R}^2$ with

$$d(P,Q) = |\bar{P} - \bar{Q}|,\ d(Q,R) = |\bar{Q} - \bar{R}|,\ d(R,P) = |\bar{R} - \bar{P}|,$$

we have

$$d(P, Q_{1/2}) \leq |\bar{P} - \bar{Q}_{1/2}|$$

where $Q_{1/2}$ is the midpoint of $\gamma_{QR}$ and $\bar{Q}_{1/2}$ is the midpoint of the line segment between $\bar{R}$ and $\bar{Q}$. The more general notion of curvature bounded from above by some constant $\kappa$ is defined by replacing $\mathbf{R}^2$ with surface $S_\kappa$ of constant curvature $\kappa$ and requiring that $\text{diam}(\triangle PQR) \leq \frac{2\pi}{\sqrt{\kappa}}$ if $\kappa > 0$. (See [ABN] for equivalent definitions.) The following important result is due to Y.G. Reshetnyak [R2]:

**Theorem 2.1** *Let $(X,d)$ be a metric space of curvature bounded from above by $\kappa$ and $\Gamma$ be a closed rectifiable curve in $X$. Then there exists a convex domain $V$ in $S_\kappa$ and a map $\varphi : V \to X$ such that $\varphi(\partial V) = \Gamma$, the lengths of the corresponding arcs coincide, and $d_{S_\kappa}(x,y) \geq d(\varphi(x), \varphi(y))$, for $x, y \in V$.*

For $O$ in $X$, we define a notion of angle for geodesics emanating from $O$. Such a definition is used in [ABN] to define a notion of tangent cone (which generalizes tangent spaces of Riemannian manifolds) in metric spaces of curvature bounded from above. The definitions below are special cases of their construction.

**Definition 2.2** *Let $\gamma$, $\sigma$ geodesics in $X$ emanating from point $P$. Then*

$$\alpha(\gamma,\sigma)=\lim_{t,s\to 0}\arccos\frac{d^2(\gamma(t),P)+d^2(\sigma(s),P)-d^2(\gamma(t),\sigma(s))}{2d(\gamma(t),P)d(\sigma(s),P)}$$

*is called the angle between $\gamma$ and $\sigma$.*

**Remark:** The limit above exists because $X$ is an NPC space.

**Definition 2.3** *Let $\Lambda_O(X)$ be the set of all geodesics emanating from the point $O$ and define an equivalence relation $\gamma\sim\sigma$ if $\alpha(\gamma,\sigma)=0$. Then $\Omega_O(X)=\Lambda_O(X)/\sim$ is called the space of directions. We denote by $\Pi:\Lambda_O(X)\to\Omega_O(X)$ the canonical projection and let $\tilde{\alpha}$ be a distance function in $\Omega_O(X)$ defined by pushing forward $\alpha$ by $\Pi$. The tangent cone $X_O$ is the cone over $\Omega_O(X)$, i.e. $X_O$ is the set $\Omega_O(X)\times[0,\infty)$ identifying all points with zero second coordinate as the point $\mathcal{O}$. The distance function $d_O$ on $X_O$ is defined by*

$$d_O^2(x,y)=t^2+s^2-2ts\cos\tilde{\alpha}(\xi_1,\xi_2) \tag{1}$$

*for $x=(\xi_1,t)$, $y=(\xi_2,s)\in\Omega_O(X)\times[0,\infty)$.*

The following lemmas are straightforward consequences of the NPC condition. (For more details see [ABN], where many of the properties of metric spaces are deduced from first principles.) We let $P,Q,R\in X$.

**Lemma 2.4** *For $\hat{Q}\in\gamma_{PQ}$ and $\hat{R}\in\gamma_{PR}$ such that $t=d_{P\hat{Q}}=d_{P\hat{R}}$, $td(\hat{Q},\hat{R})\le d(Q,R)$.*

**Lemma 2.5** *Two geodesics emanating from a point diverge in comparison to Euclidean rays of the same angle. In other words, if $\alpha_0=\alpha(\gamma_{PQ},\gamma_{PR})$, then*

$$d_{RQ}^2\ge d_{PQ}^2+d_{PR}^2-2d_{PQ}d_{PR}\cos\alpha_0.$$

**Lemma 2.6** *Geodesics in an NPC space are continuously dependent on their endpoints.*

In particular, this implies that NPC spaces are simply connected.

*Variational Theory in Complete Metric Spaces*

Let $\Omega$ be a compact domain in $\mathbf{R}^n$ and $(X,d)$ any complete metric space. In [KS1], Korevaar and Schoen develop the space $W^{1,2}(\Omega, X)$. (In [KS1], $\Omega$ is allowed to be a Riemannian domain, but an Euclidean domain is sufficient for our purposes.) Here we define this space and collect some of their results.

A Borel measurable map $u : \Omega \to X$ is said to be in $L^2(\Omega, X)$ if for $P \in X$,

$$\int_\Omega d^2(u(x), P)dz < \infty.$$

Note that by the triangle inequality, this definition is independent of $P$ chosen. For $u \in L^2(\Omega, X)$, we can construct an $\epsilon$ approximate energy function $e_\epsilon : \Omega_\epsilon \to \mathbf{R}$,

$$e_\epsilon(x) = n|\partial B_\epsilon(x)|^{-1} \int_{\partial B_\epsilon(x)} \frac{d^2(u(x), u(y))}{\epsilon^2} d\Sigma.$$

Here $\Omega_\epsilon$ is the set of points in $\Omega$ with distance from the boundary more than $\epsilon$ and $B_\epsilon(x)$ is a ball of radius $\epsilon$ centered at $x$. Letting $e_\epsilon(x) = 0$ for $\Omega - \Omega_\epsilon$, we have that $e_\epsilon(x) \in L^1(\Omega)$ and by integrating against continuous functions with compact support, these functions define linear functionals $E_\epsilon : C_c(\Omega) \to \mathbf{R}$. We say $u \in L^2(\Omega, X)$ has finite energy (or that $u \in W^{1,2}(\Omega, X)$) if

$$E^u \equiv \sup_{f \in C_c(\Omega), 0 \leq f \leq 1} \limsup_{\epsilon \to 0} E_\epsilon(f) < \infty.$$

It can be shown that if $u$ has finite energy, the measures $e_\epsilon(x)dx$ converge in the weak* topology to a measure which is absolutely continuous with respect to the Lebesgue measure. Hence, there exists a function $e(x)$, which we call the energy density, so that $e_\epsilon(x)dx \rightharpoonup e(x)dx$. In analogy to the case of real valued functions, we write $|\nabla u|^2(x)$ in place of $e(x)$. In particular,

$$E^u = \int_\Omega |\nabla u|^2 dx.$$

Similarly, the directional energy measures $|u_*(Z)|^2 dx$ for $Z \in \Gamma\bar{\Omega}$ can also be defined as the weak* limit of measures ${}^Z e_\epsilon dx$, where

$${}^Z e_\epsilon(x) = \frac{d^2(u(x), u(x + \epsilon Z))}{\epsilon^2}.$$

Furthermore, for $Z \in T\bar{\Omega}$,

$$|u_*(Z)|(x) = \lim_{\epsilon \to 0} \frac{d(u(x), u(x + \epsilon\omega)}{\epsilon},$$

a.e. $x \in \Omega$. Finally, we have

$$|\nabla u|^2 = \int_{S^{n-1}} |u_*(Z)|^2 d\sigma(Z).$$

This definition of Sobolev space $W^{1,2}(\Omega, X)$ is consistent with the usual definition when $X$ is a Riemannian manifold.

The following is the solution to the Dirichlet Problem in this setting.

**Theorem 2.7 ([KS1], [S1], [S2])** *Let $\Omega$ be a Lipschitz domain, $(X, d)$ be a complete metric space of curvature bounded from above by $\kappa$ and $\phi \in W^{1,2}(\Omega, X)$. If $\kappa > 0$, assume further that $\phi(\Omega)$ lies within a sufficiently small geodesic ball in $X$. Define*

$$W^{1,2}_{\phi} = \{u \in W^{1,2}(\Omega, X) |\, \mathrm{tr}(u) = \mathrm{tr}(\phi)\}.$$

*Then there exists $u$ such that*

$$E(u) = \inf_{v \in W^{1,2}_{\phi}} E(v).$$

*Furthermore, $u$ is locally Lipschitz continuous in the interior and Hölder continuous up to the boundary.*

The regularity of the map $u$ above is due to Korevaar and Schoen in the case $\kappa \leq 0$ and due to Serbinowski (using the assumption that $\phi(\Omega)$ lies in a small geodesic ball) in the case $\kappa > 0$.

Furthermore, (still assuming an upper curvature bound of $\kappa$) we can make sense of the notion of the pull back inner product

$$\pi : \Gamma(T\bar{\Omega}) \times \Gamma(T\bar{\Omega}) \to L^1(\bar{\Omega})$$

for any map $u \in W^{1,2}(\Omega, X)$ defined by

$$\pi(V, W) = \frac{1}{4}|u_*(V + W)|^2 - \frac{1}{4}|u_*(V - W)|^2 \text{ for } V, W \in \Gamma(T\bar{\Omega}).$$

The construction of $\pi$ above is due to Korevaar and Schoen [KS1] in the case $\kappa \leq 0$ and the author [M] in the case $\kappa > 0$. Hence, for $u \in W^{1,2}(D, X)$ where $D$ is the unit disk in the plane, we can define the area as

$$A(u) = \int_D \sqrt{det(\pi)}dz.$$

Using the variational tools developed in [KS1] and using classical arguments, one can solve the Plateau Problem in this setting (see [M]) and we have:

**The Plateau Problem** *Let $\Gamma$ be a closed Jordan curve in $X$ (where $X$ is a metric space of curvature bounded from above by $\kappa$) and let*

$$C_\Gamma = \{u \in W^{1,2}(D, X) : u|_{\partial D} \textit{ parameterizes } \Gamma \textit{ monotonically}\}.$$

*There exists $u \in C_\Gamma$ so that $A(u) = \inf\{A(v)|v \in C_\Gamma\}$. Moreover, $u$ is weakly conformal, i.e. $\pi_{11} = \pi_{22}$ and $\pi_{12} = 0 = \pi_{21}$ and Lipschitz continuous in the interior of $D$ and continuous up to the boundary.*

We call $\lambda = \pi_{11}$ the conformal factor of the pull back inner product. We have the following in [M]:

**Theorem 2.8** *The function $\lambda \in L^1(D)$ defined above is actually in $H^1_{loc}(D)$ and satisfies*

$$\int_D \log \lambda \triangle \phi \geq -\kappa \int_D \lambda \phi$$

*for any $\phi \in C_c^\infty(D)$.*

In particular, using the properties of subharmonic functions, the above theorem implies that $\lambda(z) > 0$ for a.e $z \in D$. In fact, in [M] we show:

**Theorem 2.9** *Let $u \in W^{1,2}(D,X)$ be a conformal energy minimizing map with conformal factor $\tilde{\lambda}$. Then $\tilde{\lambda} > 0$ for a.e. $z \in D$. Furthermore, there is a representative $\lambda$ in the $L^1$-equivalence class of the function $\tilde{\lambda}$ defined by*

$$\lambda(z) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{D_r(z)} \tilde{\lambda}(x) dx$$

*so that the perimeter of the set $\{z : \lambda(z) < t\}$ goes to zero as $t \to 0$.*

# 3 The Plateau Problem

¿From this point on in the paper, we will assume the hypothesis of Theorem 1.2, and in particular, $\dim X = 2$. Let $P \in X$ have cone length $r$. The map $u$ corresponding to $P$ in Theorem 1.2 will be the solution to a Plateau Problem. In order to set up this Plateau Problem, we will first construct a simple closed curve $\Gamma$ of finite length. (Note that $\tilde{\Gamma} = \partial B_r(P)$ need not be of finite length.) For this $\Gamma$, Lemma 3.3 says that the solution $u : D \to X$ of the Plateau Problem has the property that $P \in u(D)$.

**Lemma 3.1** *Let $Q_1, Q_2 \in \tilde{\Gamma}$. Let $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be the two components of $\tilde{\Gamma} - \{Q_1, Q_2\}$. For all $\delta > 0$, there exists $\epsilon > 0$ with the following property: For any $Q_1, Q_2 \in \tilde{\Gamma}$ and components $\tilde{\Gamma}_1, \tilde{\Gamma}_2$ of $\tilde{\Gamma} - \{Q_1, Q_2\}$, if $Q \in \tilde{\Gamma}_1$ with $d(Q, Q_1), d(Q, Q_2) \geq \delta > 0$, then $d(Q, \tilde{\Gamma}_2) > \epsilon$.*

*Proof:* Suppose the statement of the lemma is not true. Then there exist sequences $Q_{1,j}, Q_{2,j} \in \tilde{\Gamma}$, components $\tilde{\Gamma}_{1,j}$ and $\tilde{\Gamma}_{2,j}$ of $\tilde{\Gamma} - \{Q_{1,j}, Q_{2,j}\}$ and sequence $Q_j \in \tilde{\Gamma}_{1,j}$ such that $d(Q_j, Q_{1,j}), d(Q_j, Q_{2,j}) \geq \delta$ and $d(Q_j, \tilde{\Gamma}_{2,j}) \to 0$. We may assume that $d(Q_{1,j}, Q_{2,j}) \geq \kappa > 0$ since $\tilde{\Gamma}$ is a simple closed curve and $d(Q_{1,j}, Q_{2,j}) \to 0$ implies that $\mathrm{diam}(\Gamma_{1,j}) \to 0$. Since $\tilde{\Gamma}$ is compact, we may choose subsequence $j' \to \infty$ such that $Q_{1,j'} \to Q_1$ and $Q_{2,j'} \to Q_2$ and $Q_{j'} \to Q \in \tilde{\Gamma}_1$ where $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are the components of $\tilde{\Gamma} - \{Q_1, Q_2\}$. But this implies $d(Q, \tilde{\Gamma}_2) = 0$, contradicting the fact that $\tilde{\Gamma}$ is a simple curve. □

Given $\epsilon > 0$, $\{B_\epsilon(Q)\}_{Q \in \tilde{\Gamma}}$ form an open covering of $\tilde{\Gamma}$. Since $\tilde{\Gamma}$ is compact, there exists a finite set $\mathcal{Q}$ such that $\{B_\epsilon(Q)\}_{Q \in \mathcal{Q}}$ covers $\tilde{\Gamma}$. There is a natural ordering of elements in

$\mathcal{Q}$ defined by a choice of a parameterization $\tilde{\gamma} : [0, T] \to X$ of the simple closed curve $\tilde{\Gamma}$, i.e. we order $\mathcal{Q}= \{Q_0, Q_1, ..., Q_{N-1}\}$ so that $t_0 < t_2 < ... < t_{N-1}$ where $t_i = \tilde{\gamma}^{-1}(Q_i)$. In what follows, the index $i$ will be denoted modulo $N$.

**Lemma 3.2** *Let $\epsilon_0$ be the $\epsilon$ corresponding to $\delta = \frac{r}{4}$ in Lemma 3.1. Let $\mathcal{Q}$ be as above with choice of $\epsilon = \min\{\epsilon_0, \frac{r}{2}\}$. Then $d(Q_i, Q) < \frac{r}{2}$ for each $Q = \tilde{\gamma}(t)$, $t \in [t_i, t_{i+1}]$.*

*Proof:* Suppose $d(Q_i, Q) \geq \frac{r}{2}$ for $Q = \tilde{\gamma}^{-1}(t)$, $t \in [t_i, t_{i+1}]$. Then there exists $t'$ so that $t_i < t' < t$, $Q' = \tilde{\gamma}(t')$ and $d(Q_i, Q'), d(Q, Q') \geq \frac{r}{4}$. By Lemma 3.1, $d(Q', \tilde{\Gamma}_2) > \epsilon$ where $\tilde{\Gamma}_2$ is the component of $\tilde{\Gamma} - \{Q_i, Q\}$ that does not contain $Q'$. Hence $Q'$ is not contained in $B_\epsilon(Q_j)$ for all $Q_j \in \mathcal{Q}$. This is a contradiction. □

Now let $\sigma_i$ be the geodesic from $Q_i$ to $Q_{i+1}$. The above lemma tells us that each point of $\sigma_i$ is at a distance less than $\frac{r}{2}$ from $Q_i$ since the distance from $\tilde{\Gamma}$ to $P$ is equal to $r$. This combined with Lemma 2.6 tells us that $\tau_i = \sigma_i \bigcup \tilde{\gamma}([t_i, t_{i+1}])$ is contractible to $Q_i$ in $X - \{P\}$. In other words, $\sigma_i$ and $\tilde{\gamma}$ restricted to $[t_1, t_{i+1}]$ are homotopically equivalent in $X - \{P\}$. Hence $\bar{\Gamma}$, the closed curve defined by taking the union of geodesics $\sigma_i$, is homotopically equivalent to $\tilde{\gamma}$. Furthermore, it is of finite length since it is a finite union of geodesics. Let $\bar{\gamma} : S^1 \to X$ be a parameterization of $\bar{\Gamma}$ and let $\xi_t : [0, 1] \to X$ be the arclength proportional parameterization of the geodesic from $P$ to $\bar{\gamma}(t)$. Let $\gamma(t) = \xi_t(r_0)$ where $r_0 > 0$ is chosen so that for each $t$, $d(P, \xi_t(r_0))$ is less than the cone length at $P$. The fact that $\gamma$ is continuous and has finite length follows directly from Lemma 2.4. By construction, it is also clear that $\gamma(S^1)$ is homotopically equivalent to $\bar{\Gamma}$ and hence non-trivial in $X - \{P\}$. Thus, we can choose $\Gamma \subset \gamma(S^1)$ which is simple, closed and non-trivial in $X - \{P\}$. By the local cone structure of $X$, it is easy to see that $\Gamma$ bounds a topological disk containing $P$ which is just the union of geodesics from $P$ to points on $\Gamma$. We will call this surface $\mathcal{C}(\Gamma)$. We will show in Section 5 that:

**Lemma 3.3** *For the Plateau solution $u : D \to X$ with boundary data $\Gamma$, $u(D) = \mathcal{C}(\Gamma)$. In particular, $P \in \mathcal{C}(\Gamma)$.*

Before we do this, we need to analyze the local behavior of conformal energy minimizing maps.

# 4 The Order Function

In this section, we use ideas developed in Part I, Sections 1 through 3 of [GS] for energy minimizing maps into non-positively curved Riemannian simplicial complexes. Here, $u$ is assumed to be a conformal energy minimizer from the disk $D$ into $X$ as in Theorem 1.2 and $\lambda$ is the conformal factor of the pull back inner product under $u$. The following is essentially Lemma 1.3 of [GS] adapted to our setting.

**Lemma 4.1** *For almost every $z \in D$, we have*

$$\lim_{\sigma \to 0} \frac{\sigma \int_{D_\sigma(z)} |\nabla u|^2}{\int_{\partial D_\sigma(z)} d^2(u(x), u(z))} = 1.$$

*Here, $D_\sigma(z) \subset D$ denotes the disk of radius $\sigma$ centered at $z$.*

*Proof:* By Theorem 1.9.6 and equation (1.10v) of [KS1], for a.e. $z \in D$, we have that

$$\lambda(z) = \lim_{\sigma \to 0} \frac{(2\pi\sigma)^{-1} \int_{\partial D_\sigma(z)} d^2(u(x), u(z))}{\sigma^2}.$$

Furthermore, since $\lambda$ is a $L^1$ function, for a.e. $z \in D$ by the Lebesgue Point Lemma,

$$\begin{aligned} \lambda(z) &= \lim_{\sigma \to 0} \frac{1}{\pi\sigma^2} \int_{D_\sigma(z)} \lambda \\ &= \frac{1}{2} \lim_{\sigma \to 0} \frac{1}{\pi\sigma^2} \int_{D_\sigma(z)} |\nabla u|^2. \end{aligned}$$

Since for a.e. $z \in D$, $\lambda(z) > 0$, the conclusion follows immediately. □

Following the notation of [GS], we let

$$Ord^u(z, \sigma, Q) = \frac{\sigma \int_{D_\sigma(z)} |\nabla u|^2}{\int_{\partial D_\sigma(z)} d^2(u(x), Q)}$$

and

$$Ord^u(z) = \lim_{\sigma \to 0} Ord^u(z, \sigma, Q_{z,\sigma}),$$

where $Q_{z,\sigma}$ is the unique minimum point of function $Q \mapsto \int_{\partial D_{\sigma(z)}} d^2(u, Q)$. The limit exists because the function $\sigma \mapsto Ord^u(z, \sigma, Q_{z,\sigma})$ is monotone increasing in $\sigma$. Furthermore, $z \mapsto Ord^u(z, \sigma, Q_{z,\sigma})$ is continuous and hence $z \mapsto Ord^u(z)$ is an upper semicontinuous function. (See [GS, Section 1.2] for more details.) This (combined with Lemma 4.1) gives $Ord(z) \geq 1$ for all $z \in D$.

Using the order function above, we will analyze the behavior of the map $u$ near a point $z \in D$. For now, we will analyze this at the origin $0 \in D$ and let $O = u(0)$. Consider the blow up maps $u_\sigma : D \to (X, d_{\mu(\sigma)})$ defined by $u_\sigma(z) = u(\sigma z)$. Here the distance function $d_{\mu(\sigma)}$ is defined by

$$d_{\mu(\sigma)}(P, Q) = \mu(\sigma)^{-1} d(P, Q)$$

with

$$\mu(\sigma) = \sqrt{\sigma^{-1} \int_{\partial D_\sigma(0)} d^2(u, O)}.$$

Note that

$$\int_{D_\kappa(0)} |\nabla u_\sigma|^2 = \mu(\sigma)^{-2} \int_{D_{\sigma\kappa}(0)} |\nabla u|^2$$

and

$$\int_{\partial D_\kappa(0)} d^2_{\mu(\sigma)}(u_\sigma, O) = \mu(\sigma)^{-2}\sigma^{-1} \int_{\partial D_{\sigma\kappa}(0)} d^2(u, O).$$

Hence,

$$\int_{\partial D_1(0)} d^2_{\mu(\sigma)}(u_\sigma, O) = \mu(\sigma)^{-2}\sigma^{-1} \int_{\partial D_\sigma} d^2(u, O) = 1$$

and

$$Ord^{u_\sigma}(0, 1, O) = Ord^u(0, \sigma, O).$$

Let $\alpha = Ord^u(0) = \lim_{\sigma\to 0} Ord^u(0, \sigma, Q_{0,\sigma})$. Since $Q_{0,\sigma} \to u(0) = O$ as $\sigma \to 0$, for sufficiently small $\sigma$, we have

$$Ord^u(0, \sigma, O) \leq 2\alpha,$$

i.e.

$$Ord^{u_\sigma}(0, 1, O) \leq 2\alpha.$$

Hence

$$\int_{D_1(0)} |\nabla u_\sigma|^2 \leq 2\alpha,$$

and $u_\sigma$ restricted to $D_r$ for $r < 1$ is uniformly Lipschitz in the sense that for every $x, y \in D_r$, there exists constant $L$ such that for sufficiently small $\sigma$

$$d_{\mu(\sigma)}(u_\sigma(x), u_\sigma(y)) \leq L|x - y|.$$

The above follows directly from the fact that energy minimizers into NPC spaces are Lipschitz away from the boundary with Lipschitz constant depending on total energy and distance to the boundary(see [KS1, Theorem 2.4.6]). Using this fact, we will obtain a homogeneous approximating map which we will use to analyze $u$ near 0.

# 5 The Homogeneous Approximating Maps

Our goal is to obtain a homogeneous approximating map to $u : D \to X$. First, we need the following construction.

Consider distance functions $d_\lambda$ on the space $X$ defined by $d_\lambda(x, y) = \lambda^{-1}d(x, y)$ for $\lambda > 0$. Note that $(X, d_\lambda)$ is still an NPC space. We let $B^\lambda_r(O)$ denote the closed geodesic ball in $X$ of radius $r$ about $O \in X$ with respect to the distance function $d_\lambda$ and $B^*_r(\mathcal{O})$ be the closed ball of radius $r$ about the origin $\mathcal{O}$ in the tangent cone $(X_O, d_O)$. We have the following:

**Lemma 5.1** *The metric spaces $(B^\lambda_R(O), d_\lambda)$ converge in Hausdorff distance to the metric space $(B^*_R(\mathcal{O}), d_O)$.*

*Proof:* It is enough to show that for every $\epsilon > 0$, there exist $\epsilon$-nets $N_R^\lambda$ and $N_R$ of $B_R^\lambda(O)$ and $B_R^*(\mathcal{O})$ such that $N_R^\lambda$ converge to $N_R$ in the Lipschitz distance as $\lambda \to 0$ (see [P] and [G]). For $0 = r_0 < r_1 < r_2 < ... < r_n = R$ with $r_{i+1} - r_i < \epsilon$, the union of the $\frac{\epsilon}{2}$-nets of $\partial B_{r_i}(O)$ forms a $\epsilon$-net of $B_R(O)$ as can be seen by the following claim.

**Claim** The interior of a quadrilateral in $\Sigma$ with side lengths less than $\epsilon$ is covered by $\epsilon$-balls centered at the vertices.

*Proof:* The statement of the claim is true if $\Sigma = \mathbf{R}^2$. Hence the claim follows from Theorem 2.1. $\Box$(*claim*)

Hence it is sufficient to find $\epsilon$-nets $N_\lambda$ of $\partial B_1^\lambda(O)$ converging in the Lipschitz distance to $\epsilon$-net $N$ of $\partial B_1^*(\mathcal{O})$. To do this, we first let $N$ be an $\frac{\epsilon}{2}$-net of $\partial B_1^*(\mathcal{O})$. $\partial B_1^*(\mathcal{O})$ is compact and we can choose $N$ which is finite, say $\{(\gamma_1, 1), ..., (\gamma_n, 1)\}$. By definition,

$$d_O((\gamma_i, 1), (\gamma_j, 1)) = \sqrt{2 - 2\cos\alpha(\gamma_i, \gamma_j)}$$

and since

$$\cos\alpha(\gamma_i, \gamma_j) = \lim_{\lambda\to 0} \frac{2\lambda^2 - d^2(\gamma_i(\lambda), \gamma_j(\lambda))}{2\lambda^2},$$

we have

$$d_O((\gamma_i, 1), (\gamma_j, 1)) = \lim_{\lambda\to 0} \lambda^{-1} d(\gamma_i(\lambda), \gamma_j(\lambda)). \tag{2}$$

Hence for sufficiently small $\lambda$, $N_\lambda = \{\gamma_i(\lambda)\}_{i=1}^n$ forms an $\epsilon$-net of $\partial B_1^\lambda(O)$. Consider $f_\lambda : (N_\lambda, d_\lambda) \to (N, d_O)$ defined by $\gamma_i(\lambda) \mapsto (\gamma_i, 1)$. By the NPC condition,

$$d_O((\gamma_i, 1), (\gamma_j, 1)) \leq d_\lambda(\gamma_i(\lambda), \gamma_j(\lambda)).$$

Furthermore, by equation 2,

$$d_\lambda(\gamma_i(\lambda), \gamma_j(\lambda)) \leq d_O((\gamma_i, 1), (\gamma_j, 1)) + 0(\lambda).$$

Letting

$$m = \min_{i\neq j} d((\gamma_i, 1), (\gamma_j, 1)),$$

we see that

$$1 \leq \frac{d_\lambda(\gamma_i(\lambda), \gamma_j(\lambda))}{d_O((\gamma_i, 1), (\gamma_j, 1))} \leq 1 + \frac{0(\lambda)}{m}.$$

This shows that $N_\lambda$ converges to $N$ in the Lipschitz distance. $\Box$

**Lemma 5.2** *Suppose $(X_k, d_k)$ are compact metric spaces converging in Hausdorff distance to a compact metric space $(X, d)$. Let $f_k : \bar{D} \to (X_k, d_k)$ be Lipschitz functions with Lipschitz constant $L$ and assume $d_k(f_k(\cdot), f_k(\cdot))$ converge uniformly to a pseudo distance $d_0(\cdot,\cdot)$ in $\bar{D}$. Then there exists $f : \bar{D} \to X$ such that $d(f(x), f(y)) = d_0(x, y)$.*

*Proof:* Let $d_H(\cdot,\cdot)$ denote the Hausdorff distance and without the loss of generality, assume $d_H(X_k, X) \leq \frac{1}{k}$ and $|d_k(f_k(\cdot), f_k(\cdot)) - d_0(\cdot,\cdot)| \leq \frac{1}{k}$. For each $k$, there exist metric space $(Z_k, d_{Z_k})$ and isometries $\psi_k : X \to Z_k$ and $\phi_k : X_k \to Z_k$ such that

$$\psi_k(X) \subset (\phi_k(X_k))_{\frac{2}{k}} \tag{3}$$

and

$$\phi_k(X_k) \subset (\psi_k(X))_{\frac{2}{k}}.$$

Here the notation $(Y)_\epsilon$ is used to denote the $\epsilon$-neighborhood of the set $Y$. For $x \in D$, let $P_{x,k}$ be the point such that

$$d_{Z_k}(P_{x,k}, \phi_k(f_k(x))) = \inf_{Q \in \psi_k(X)} d_{Z_k}(Q, \phi_k(f_k(x))).$$

Note that $P_{x,k} \in \psi_k(X)$ since $X$ is compact and $\psi_k$ is an isometry and, furthermore,

$$d_{Z_k}(P_{x,k}, \phi_k(f_k(x))) \leq \frac{2}{k}$$

by equation 3. Let $Q_{x,k} = \psi^{-1}(P_{x,k}) \in X$. Since $X$ is compact, there exists a subsequence $Q_{x,k'}$ converging to point $Q_x$.

Let $C = \{x_n\}$ be a countable dense subset of $D$. By the diagonalization process, we can extract sequence $k_1 < k_2 < \ldots$ so that for each $n$, $Q_{x_n,k_j}$ converge to $Q_{x_n}$. Furthermore,

$$\begin{aligned}
d(Q_{x_n}, Q_{x_m}) &= \lim_{j\to\infty} d(Q_{x_n,k_j}, Q_{x_m,k_j}) \\
&= \lim_{j\to\infty} d_{Z_{k_j}}(P_{x_n,k_j}, P_{x_m,k_j}) \\
&= \lim_{j\to\infty} d_{Z_{k_j}}(\phi_{k_j}(f_{k_j}(x_n)), \phi_{k_j}(f_{k_j}(x_m))) \\
&= \lim_{j\to\infty} d_{k_j}(f_{k_j}(x_n), f_{k_j}(x_m)) \qquad (4) \\
&= d_0(x_n, x_m)
\end{aligned}$$

Define $f : \bar{D} \to X$ in the following way: first for $x \in C$, let $f(x) = Q_x$. Now note that any $y, z \in C$, equation 4 and the fact that the Lipschitz constant for $f_k$ is $L$ independently of $k$ shows that $d(Q_y, Q_z) \leq L|y - z|$. Hence for $x \in D$ and a subsequence $\{x_i\} \subset C$ converging to $x$, $Q_{x_i}$ converges, say to a point $Q$. We define $f(x) = Q$. By construction, $f$ has the desired properties. □

We follow the notation of Section 4 and let $(r, \theta)$ be the standard polar coordinates of $D_r$. Let $\sigma_i$ be a sequence converging to 0. Fix $r < 1$ and consider pseudo distance functions $d_{\sigma_i}$ on $\bar{D}_r$ defined by pulling back the distance functions $d_{\mu(\sigma_i)}$ under $u_{\sigma_i}$. We claim that the functions

$$d_{\sigma_i} : \bar{D}_r \times \bar{D}_r \to \mathbf{R}$$

are uniformly Lipschitz (independent of $i$) with respect to the product topology on $\bar{D}_r \times \bar{D}_r$. This follows easily from the triangle inequality and the fact that $\{u_{\sigma_i}\}$ is uniformly Lipschitz: let $(x_1, x_2), (y_1, y_2) \in \bar{D}_r \times \bar{D}_r$, then

$$\begin{aligned} d_{\sigma_i}(x_1, x_2) - d_{\sigma_i}(y_1, y_2) &= |d_{\mu(\sigma_i)}(u_{\sigma_i}(x_1), u_{\sigma_i}(x_2)) - d_{\mu(\sigma_i)}(u_{\sigma_i}(y_1), u_{\sigma_i}(y_2)| \\ &\leq d_{\mu(\sigma_i)}(u_{\sigma_i}(x_1), u_{\sigma_i}(y_1)) + d_{\mu(\sigma_i)}(u_{\sigma_i}(x_2), u_{\sigma_i}(y_2)) \\ &\leq L|x_1 - y_1| + L|x_2 - y_2|. \end{aligned}$$

Hence there is a subsequence of $\{d_{\sigma_i}\}$, which we still call $\{d_{\sigma_i}\}$, that converges uniformly to a pseudo distance function $d_\infty$ defined on $\bar{D}_r$. From Lemmas 5.1 and 5.2, there is a map $u_0 : \bar{D}_r \to (X_0, d_O)$ such that $d_O(u_0(\cdot), u_0(\cdot)) = d_\infty(\cdot, \cdot)$. We call $u_0$ a homogeneous approximating map and justify this terminology as follows. Notice, by construction, we have

$$\begin{aligned} Ord^{u_{\sigma_i}}(0, \kappa, O) &= \frac{\kappa \int_{D_\kappa(0)} |\nabla u_{\sigma_i}|^2}{\int_{\partial D_\kappa(0)} d^2_{\mu(\sigma_i)}(u_{\sigma_i}(x), O)} \\ &= \frac{\sigma_i \kappa \int_{D_{\sigma_i \kappa}(0)} |\nabla u|^2}{\int_{\partial D_{\sigma_i \kappa}(0)} d^2(u(x), O)}. \end{aligned}$$

Hence, letting $\sigma_i \to 0$ (and using $\lim_{i\to\infty} Q_{0,\sigma_i} = O$ on the left hand side), we get that for all $\kappa > 0$,

$$Ord^{u_0}(0, \kappa, O) = Ord^u(0) = \alpha \geq 1.$$

By Lemma 3.2, [GS], $u_0 : D_r \to (X_O, d_O)$ is intrinsically homogeneous, i.e.

$$d_O(u_0(x), u_0(0)) = \left(\frac{|x|}{r}\right)^\alpha d_O(u_0(\frac{rx}{|x|}), u_0(0)). \tag{5}$$

By Theorem 3.11 of [KS2], $u_0$ is a conformal map and hence

$$d_O(u_0(x), u_0(0)) = |x|^\alpha \beta$$

for some constant $\beta$. Furthermore (again by [GS]) the image of the curve $t \to u(tx)$, $0 \leq t \leq r$, is a geodesic for any $x \in \partial D_r$ and $d_O(u_0(\cdot), u_0(\cdot))$ defines a cone metric on $\bar{D}_r$. This shows that the image of $\partial D_r$ under $u_0$ is a closed curve (and not a segment). By the cone structure and the two dimensionality of $X_0$, this implies that $u_0(\partial D_r)$ is non-trivial in $X_0 - \{\mathcal{O}\}$. Therefore:

**Lemma 5.3** *There exists $\delta_0 > 0$ such that for all $\theta \in [0, 2\pi]$, $u_0(r, \theta) \neq u_0(r, \theta + \delta_0)$.*

*Proof:* This follows from the fact that the length of a closed loop in $X_0 - \{\mathcal{O}\}$ at a distance $r$ away is at least $2\pi r$ by the NPC condition. $\Box$.

We now wish to show:

**Lemma 5.4** *There exists $\sigma_i$ such that $u_{\sigma_i}(\partial D_r)$ is homotopically non-trivial in $X - \{u_{\sigma_i}(0)\}$.*

*Proof:* Let $\gamma_i(\cdot) = u_{\sigma_i}(r, \cdot)$. Suppose for every $i$, $\gamma_i$ is trivial. Fix $\delta_0 > 0$ as in Lemma 5.3. By the local cone structure of $X$, for all $i$ sufficiently large, there exists $\theta_i$ such that $\gamma_i(\theta_i)$ is a point on the geodesic from $u_{\sigma_i}(0)$ to $\gamma_i(\theta_i + \delta_0)$ or $\gamma_i(\theta_i + \delta_0)$ is a point on the geodesic from $u_{\sigma_i}(0)$ to $\gamma_i(\theta_i)$. Choose a subsequence which we still denote $\{\theta_i\}$ such that $\theta_i \to \theta_0$. Recall that $d_{\sigma_i}$ converges to $d_0$; in particular

$$|d_{\sigma_i}((r, \theta_i), 0)) - d_{\sigma_i}((r, \theta_i + \delta_0), 0)|$$

converges to

$$|d_0(u_0(r, \theta_0), u_0(0)) - d_0(u_0(r, \theta_0 + \delta_0), u_0(0))|.$$

as $i \to \infty$ and therefore

$$|d_{\sigma_i}((r, \theta_i), 0)) - d_{\sigma_i}((r, \theta_i + \delta_0), 0)| \to 0.$$

Hence we see that

$$d_{\sigma_i}((r, \theta_i), (r, \theta_i + \delta_0)) \to 0$$

and this implies

$$d_0(u_0(r, \theta_0), u_0(r, \theta_0 + \delta_0)) = 0.$$

But this contradicts Lemma 5.3. $\Box$

Since we can do this analysis at any point $x \in D$, we see that:

**Lemma 5.5** *Let $u : D \to X$ be a solution to the Plateau Problem. For each $x \in D$, there exists $\sigma > 0$ sufficiently small such that $u(\partial D_\sigma(x))$ is non-trivial in $X - \{u(x)\}$.*

We can finally prove Lemma 3.3.

*Proof of Lemma 3.3:* Suppose there exists $x \in D$ such that $x$ is not in the set $\mathcal{C}(\Gamma)$. Let $\gamma$ be a geodesic from $P$ to $u(x)$. Let $\tilde{\gamma}$ be the union of all geodesic extensions of $\gamma$. $D' = \{y \in \bar{D} : u(y) \in \tilde{\gamma}\}$ is a closed set and hence there exists $x_0 \in \bar{D}$ such that

$$d(u(x_0), u(0)) = \max_{y \in D'} d(u(y), u(0)). \tag{6}$$

Since $x$ is not in $\mathcal{C}(\Gamma)$, $x_0$ is not in $\partial D$. By Lemma 5.5, there is a $\sigma > 0$ sufficiently small such that $u(\partial D_\sigma(x_0))$ is non-trivial in $X - \{u(x_0)\}$. Let $G$ be the union of geodesics from $P$ to $u(\partial D_\sigma(x_0))$. Then $u(x_0) \in G$; otherwise we contradict the fact that $u(\partial D_\sigma(x_0))$ is non-trivial in $X - \{u(x_0)\}$. Hence, there exists a geodesic from $u(0)$ to a point $Q$ on $u(\partial D_\sigma(x_0))$ that contains $u(x_0)$ in the interior and hence $d(u(x_0), u(0)) < d(Q, u(0))$. This contradicts equation 6. Hence $u(D) \subset \mathcal{C}(\Gamma)$. On the other hand, for any $Q, Q' \in \mathcal{C}(\Gamma)$, $\Gamma$ is not contractible to $Q$ in $\mathcal{C}(\Gamma) - \{Q'\}$. Hence $u(D) = \mathcal{C}(\Gamma)$. □

# 6 Proof of Injectivity

Here, we finally show that $u$ is a homeomorphism. The important fact is that in the last section, we proved $u(D) = \mathcal{C}(\Gamma)$. Let $\Sigma = \mathcal{C}(\Gamma)$ in this section.

**Lemma 6.1** *The map $u$ is a homeomorphism.*

*Proof:* Suppose that there exists $x_1, x_2 \in D$ with $x_1 \neq x_2$ so that $u(x_1) = u(x_2)$. Let $(r_1, \theta_1)$ and $(r_2, \theta_2)$ be the standard polar coordinates of $x_1$ and $x_2$. Let $\sigma_0$ be the curve in $D$ obtained by taking the union of the ray from $(r_1, \theta_1)$ to $(1, \theta_1)$, one of boundary arc of $D$ from $(1, \theta_1)$ to $(1, \theta_2)$ and, lastly, the ray from $(1, \theta_2)$ to $(r_2, \theta_2)$. Then the curve $u \circ \sigma_0$ is a closed curve. Let $R_0$ be the subset in $\Sigma$ bounded by $u \circ \sigma_0$. Since $u$ restricted to $\partial D$ is a monotone parameterization of $\Gamma$, $R_0$, is not all of $u(D)$. The portion of $u \circ \sigma_0$ that is the boundary of $R_0$ is a closed curve, hence there exists $y_1, y_2$ with $y_1 \neq y_2$ on $\sigma_0$ so that $u(y_1) = u(y_2) \in \partial R_0$. Since $u$ is a continuous map, $y_1$, $y_2$ are boundary points of $u^{-1}(\bar{R}_0)$. Choose any curve $\sigma_1$ from $y_1$ to $y_2$ which does not intersect $u^{-1}(\bar{R}_0)$ except at $y_1$ and $y_2$. Note that $u \circ \sigma_1$ is a closed curve.

By choice of $\sigma_1$, $u \circ \sigma_1$ does not intersect $\bar{R}_0$ except at $u(y_1) = u(y_2)$. Hence, by construction, the subset $R_1$ of $\Sigma$ bounded by $u \circ \sigma_1$ does not intersect $R_0$ and $\bar{R}_0 \cap \bar{R}_1 = \{Q\}$, where $Q = u(y_1) = u(y_2)$.

**Claim** There exists a connected set $C$ containing $y_1$ and $y_2$ which is mapped to $Q$.

*Proof:* Let $\Omega$ be the region in $D$ bounded by the union of curves $\sigma_0$ and $\sigma_1$. Let $z_0 \in \sigma_0$ and $z_1 \in \sigma_1$ with $z_i \neq y_j$, $i, j = 1, 2$. If there exist no connected set as in the statement of the claim, then there exists a curve $\gamma$ in $\bar{\Omega}$ so that $u \circ \gamma$ does intersect the point $Q$. But this is impossible because $\bar{\Omega}$ is mapped into $\bar{R} \cup \bar{R}_1$ as we will show. If $\bar{\Omega}$ is not mapped into $\bar{R} \cup \bar{R}_1$, then there exists $z \in \Omega - \partial D$ such that $u(z) \in \partial(\bar{R}_0 \cup \bar{R}_1)$. But there is $\sigma > 0$ sufficiently small so that $u(\partial D_\sigma(z))$ is a non-trivial curve in $\Sigma - \{u(z)\}$ by Lemma 5.5. Therefore, since $\Sigma$ is a surface, $u(z)$ cannot be a boundary of the set $R_0 \bigcup R_1$.

□

But the assertion of this claim contradicts the fact that there exists $\sigma > 0$ so that $u(\partial D_\sigma(y_1))$ is non-trivial in $\Sigma - \{u(y_1)\}$ . Hence, there cannot be $x_1, x_2 \in D$ such that $u(x_1) = u(x_2)$ unless $x_1 = x_2$. □

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