Regularity of harmonic maps from a flat complex

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Abstract. We show that a harmonic map from an admissible flat simplicial complex into a metric space of non-positive curvature is Lipschitz continuous away from the $(n-2)$-simplices.

1. Introduction

The study of harmonic maps into singular targets was initiated by the work of [**GS**] who considered locally compact polyhedral targets of non-positive curvature. In particular, these harmonic maps were shown to be Lipschitz continuous in the interior of the domain. In [**KS1**], the interior Lipschitz regularity result for harmonic maps was extended to the case when the target is a (not necessarily locally compact) metric space of non-positive curvature (which we will refer to as a NPC space). The interior Lipschitz regularity of harmonic maps from a Riemannian domain into metric spaces of curvature bounded from above with a smallness of image assumption was shown by [**Se1**].

The papers [**Ch**], [**EF**] and [**F**] studies the regularity of harmonic maps from singular domains. In particular, [**EF**] considers domain X which is an admissible Riemannian simplicial complex and a target Y which is a NPC space and proves that a harmonic map $f: X \to Y$ is pointwise Hölder continuous in the interior of the domain. The interior pointwise Hölder regularity of harmonic maps from an admissible Riemannian simplicial complex into a locally compact metric spaces of curvature bounded from above with a smallness of image assumption was shown in [**F**].

A natural question is to ask when a harmonic map from a singular domain is Lipschitz continuous. In this paper, we consider the interior regularity of harmonic maps from an admissible flat simplicial complex. We show

THEOREM 1. *Let X be a n-dimensional admissible flat simplicial complex, Y a metric space of non-positive curvature and $f: X \to Y$ a harmonic map. Then f is Lipschitz continuous away from ∂X and the $(n-2)$-simplices of X.*

Theorem 1 shows that the Lipschitz regularity of harmonic maps can be shown for a domain that is not topologically a manifold. Although we restrict ourselves to the flat metric on the domain in this paper, the Lipschitz regularity of harmonic

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maps should still hold for a general admissible n-dimensional Riemannian simplicial complex with appropriate assumptions on the metric near the $(n-1)$ simplices. We conjecture that Theorem 1 can be generalized to include the case when the smooth Riemannian metric defined on each n-dimensional simplex F of X extends smoothly beyond each $(n-1)$-dimensional simplex of F. We hope to tackle this issue in a future paper. On the other hand, the result that f is Lipschitz continuous away from $(n-2)$-simplices optimal since the Lipschitz regularity property can fail at $(n-2)$-simplices. For example, the function $f(r,\theta) = r^a \cos\theta$ on a unit disk D with metric given by $g = dr^2 + \frac{1}{a^2} r^2 d\theta^2$ in polar coordinates is a harmonic map. Clearly, when $a < 1$ (which corresponds to a (D, g) being a two-dimensional metric cone with negative curvature), f is not Lipschitz at 0.

In [**DM**], we prove the Lipschitz regularity of harmonic maps from a two dimensional admissible flat simplicial complex away from 0-simplices. The Lipschitz regularity is important in applications of harmonic map theory. For example, in [**DM**], the Lipschitz continuity was crucial in the existence and compactness theorems needed for the study of hyperbolic manifolds. The method of proof is quite different when we consider higher dimensional domains. The argument of [**DM**] uses the Hopf differential and hence is strictly a two dimensional argument.

2. Definitions

A connected locally finite n-dimensional simplicial complex is called admissible (cf. [**Ch**] and [**EF**]) if the following two conditions hold:

(i) X is dimensionally homogeneous, i.e., every simplex is contained in a n-simplex, and
(ii) X is $(n-1)$-chainable, i.e., every two n-simplices A and B can be joined by a sequence $A = F_0, e_0, F_1, e_1, ..., F_{k-1}, e_{k-1}, F_k = B$ where F_i is a n-simplex and e_i is a $(n-1)$-simplex contained in F_i and F_{i+1}.

The boundary ∂X of X is the union of all simplices of dimension $n-1$ which is contained in only one n-dimensional simplex. Here and henceforth, we use the convention that simplices are understood to be closed. A locally finite simplicial complex is called a Riemannian simplicial complex if a smooth bounded Riemannian metric is defined on n-simplex. This set of Riemannian metrics induces a distance function on X which we will denote by $d_X(\cdot,\cdot)$. We say a Riemannian simpicial complex is flat if:

(i) the Riemannian metric g_A on each n-simplex A makes (A, g_A) isometric to a subset of $\mathbf{R}^n$, and
(ii) if A and B are adjacent n-simplex sharing a $(n-1)$-simplex e, the metrics g_A and g_B induce the same metric g_e on e which makes (e, g_e) isometric to a subset of $\mathbf{R}^{n-1}$.

A complete metric space (Y, d) is said to have curvature bounded from above by κ if the following conditions are satisfied:

(i) The space (Y, d) is a length space. That is, for any two points P and Q in Y, there exists a rectifiable curve γ_{PQ} so that the length of γ_{PQ} is equal to $d(P, Q)$ (which we will sometimes denote by d_{PQ} for simplicity). We call such distance

realizing curves geodesics.
(ii) Let $a = \sqrt{|\kappa|}$. Every point $P_0 \in Y$ has a neighborhood $U \subset Y$ so that given $P, Q, R \in U$ (assume $d_{PQ} + d_{QR} + d_{RP} < \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$) with Q_t defined to be the point on the geodesic γ_{QR} satisfying $d_{QQ_t} = td_{QR}$ and $d_{Q_tR} = (1-t)d_{QR}$, we have

$$\cosh(ad_{PQ_t}) \leq \frac{\sinh((1-t)ad_{QR})}{\sinh(ad_{QR})}\cosh ad_{PQ} + \frac{\sinh(tad_{QR})}{\sinh(ad_{QR})}\cosh ad_{PR}$$

for $\kappa < 0$,

$$d^2_{PQ_t} \leq (1-t)d^2_{PQ} + td^2_{PR} - t(1-t)d^2_{QR}$$

for $\kappa = 0$, and

$$\cos(ad_{PQ_t}) \geq \frac{\sin((1-t)ad_{QR})}{\sin(ad_{QR})}\cos ad_{PQ} + \frac{\sin(tad_{QR})}{\sin(ad_{QR})}\cos ad_{PR}$$

for $\kappa > 0$.

We will say that (Y, d) is NPC (non-positively curved) if it has curvature bounded from above by 0. A simply connected space of curvature bounded from above by κ is commonly referred to as a $CAT(\kappa)$ space in literature.

A map from X into Y is called harmonic if it is locally energy minimizing. Recall that, when (X^m, g) and (Y^n, h) are Riemannian manifolds, then the energy of $f : X \to Y$ is

$$E^f := \int_X |\nabla f|^2 d\mu$$

where

$$|\nabla f|^2(x) = \sum_{\alpha,\beta=1}^{m} g^{\alpha\beta}(x)h_{ij}(f(x))\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}$$

with (x^α) and (f^i) the local coordinate systems around $x \in X$ and $f(x) \in Y$ respectively.

If (X, g) is a Riemannian manifold but (Y, d) is only assumed to be a complete metric space, then we use the Korevaar-Schoen definition of energy where E^f is defined as above with $|\nabla f|^2 d\mu$ the weak limit of ϵ-approximate energy density measures. The ϵ-approximate energy density measures are measures derived from the average difference quotients. More speficially, define $e_\epsilon : X \to \mathbf{R}$ by

$$e_\epsilon(x) = \begin{cases} \int_{S(x,\epsilon)} \frac{d^2(f(x),f(y))}{\epsilon^2}\frac{d\sigma_{x,\epsilon}}{\epsilon^{n-1}} & \text{for } x \in X_\epsilon \\ 0 & \text{for } x \in X - X_\epsilon \end{cases}$$

where $\sigma_{x,\epsilon}$ is the induced measure on the ϵ-sphere $S(x, \epsilon)$ centered at x and $X_\epsilon = \{x \in X : d(x, \partial X) > \epsilon\}$. This in turn defines a family of functionals $E^f_\epsilon : C_c(X) \to \mathbf{R}$ by setting

$$E^f_\epsilon(\varphi) = \int_X \varphi e_\epsilon d\mu.$$

We say f has finite energy (or that $f \in W^{1,2}(X, Y)$) if

$$E^f := \sup_{\varphi \in C_c(X), 0 \leq \varphi \leq 1} \limsup_{\epsilon \to 0} E^f_\epsilon(\varphi) < \infty.$$

It can be shown that if f has finite energy, the measures $e_\epsilon(x)dx$ converge weakly to a measure which is absolutely continuous with respect to the Lebesgue measure. Therefore, there exists a function $e(x)$, which we call the energy density, so that

$e_\epsilon(x)d\mu \rightharpoonup e(x)d\mu$. In analogy to the case of real valued functions, we write $|\nabla f|^2(x)$ in place of $e(x)$. In particular,

$$E^f = \int_X |\nabla f|^2 d\mu.$$

For $V \in \Gamma X$ where ΓX is the set of Lipschitz vector fields on X, the directional energy measure $|f_*(V)|^2 d\mu$ is similarly defined. The real valued L^1 function $|f_*(V)|^2$ generalizes the norm squared on the directional derivative of f.

Finally, the Korevaar-Schoen definition of energy can exteded to the case when X is an admissible Riemannian simplicial complex. Here, the energy E^f is

$$\int_X |\nabla f|^2 d\mu := \int_{\cup_{i=1}^k F_i} |\nabla f|^2 d\mu$$

where $\{F_i\}_{k=1,\dots,k}$ is the set of all top-dimensional simplices of X. The functions $|\nabla f|^2$ and $|f_*(V)|^2$ are defined for almost every point in X.

3. Summary of relevant results

The Lipschitz continuity for a harmonic map from a Riemannian domain is due to Korevaar and Schoen.

Theorem 2 ([**KS1**]). *Let (Ω, g) be a n-dimensional Lipschitz Riemannian domain, (Y, d) a NPC space and $f : \Omega \to Y$ a harmonic map. Then f is locally Lipschitz continuous in the interior of Ω, where the local Lipschitz constant is bounded above by*

$$C\left(\frac{E^f}{\min\{1, dist(x, \partial\Omega)\}}\right)^{\frac{1}{2}}$$

where C is a constant which depends only on n and the regularity of the metric g.

The boundary regularity result is due to Serbinowski.

Theorem 3 ([**Se2**]). *Let (Ω, g) be a n-dimensional Lipschitz Riemannian domain, (Y, d) a NPC space, $\phi \in W^{1,2}(\Omega, Y)$ and $f : \Omega \to Y$ a harmonic map such that $f = \phi$ on $\partial\Omega$. If, for a relatively open subset $\Gamma \subset \partial\Omega$, ϕ is locally Hölder continuous in a neighborhood of $\Gamma \cup \Omega$ with Hölder exponent α, then f is locally Hölder continuous with Hölder exponent β for every $0 \leq \beta < \alpha$.*

The Hölder regularity result is due to Eells and Fuglede.

Theorem 4 ([**EF**]). *Let X be an admissible Riemanian simplicial complex, (Y, d) a NPC space and $f : X \to Y$ a harmonic map. Then f is pointwise Hölder continuous; i.e. for any point $a \in X$, there exists posibive constants A, α, δ depending on a so that*

$$d(f(x), f(a)) \leq A d_X(x, a)^\alpha$$

whenever $d_X(x, a) < \delta$.

4. Preliminary results

We will henceforth assume that the domain of a harmonic map is a flat 2-dimensional simplicial complex for simplicity. The proof of the Lipschitz regularity result for domain dimension equal to 2 given below generalize in a straighforward manner to the case when the domain is a flat n-dimensional simplicial complex.

Since this is a local result, we consider a harmonic map $f : X_1 \to (Y, d)$ where X_1 is isometric to a neighborhood of a point of a $(n-1)$-simplex away from $(n-2)$-simplices. We define X_1 below. Take n copies of the unit upper half disk $D^+ = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 < 1, y \geq 0\}$. We would like to distinguish these copies so we label them $D_1^+, ..., D_n^+$ and use (x_i, y_i) to denote the point corresponding to $(x, y) \in D^+$ on the ith copy D_i^+. Let $X_1 = \cup_{i=1}^n D_i^+ / \sim$ where $\sim$ is defined by

$$(x_i, 0) \sim (x_j, 0) \quad \text{for } x \in \mathbf{R}. \tag{1}$$

In other words, $\sim$ identifies the x-axis of D_i^+ to the x-axis of D_j^+ for all i and j. Because of this identification, we can denote $(x_i, 0)$ by $(x, 0)$. We will denote $(0, 0)$ by 0. We let $\Gamma_r = \{(x, 0) \in X_1 : -r < x < r\}$ and $\Gamma_{r,\epsilon} = \{(x_i, y_i) \in X_1 : -r < x_i < r, 0 \leq y < \epsilon\}$. Furthermore, for $z^1 = (x_i^1, y_i^1), z^2 = (x_i^2, y_i^2) \in X_1$, we let $|z^1 - z^2|$ be the distance between z^1 and z^2. In particular, $|z| = |z - 0| = \sqrt{x_i^2 + y_i^2}$. Finally, let $X_r = \{z = (x_i, y_i) \in X_1 : |z| < r\}$ and $D_{r,i}^+ = D_i^+ \cap X_r$.

We first prove that harmonic maps from X_1 is Lipschitz continuous in the direction parallel to Γ_1.

LEMMA 5. *Let $f : X_1 \to Y$ be a harmonic map. For any $r < 1$, there exists a constant L dependent only on r and the energy E^f of f so that $d(f(z^1), f(z^2)) \leq L|z^1 - z^2|$ for $z^1 = (x_i^1, y_i^1), z^2 = (x_i^2, y_i^2) \in X_r$ provided that $y^1 = y^2$.*

PROOF. For the vector $\frac{\partial}{\partial x}$, we will denote the directional derivative measure $|f_*(\frac{\partial}{\partial x})|^2 dxdy$ by $\left|\frac{\partial f}{\partial x}\right|^2 dxdy$ for simplicity. By [**KS1**],

$$\left|\frac{\partial f}{\partial x}\right|^2 (x_i, y_i) = \lim_{\epsilon \to 0} \frac{d^2(f(x_i + \epsilon, y_i), f(x_i, y_i))}{\epsilon^2}$$

for a.e. $(x_i, y_i) \in X_1$. It will be sufficient to show an upper bound for $\left|\frac{\partial f}{\partial x}\right|^2$ in X_r which is dependent only on r and E^f.

Let D_r be a disk of radius r, take n copies of D_r and label them $D_{r,1}, ..., D_{r,n}$. Let $\Omega_r = \cup_{i=1}^n D_{r,i} / \approx$ where the equivalence relation $\approx$ is defined so that $(x_i, 0) \approx (x_j, 0)$. For a fixed $r_0 < 1$ and $\epsilon > 0$ sufficiently small so that $r_0 + \epsilon < 1$, let $g : X_{r_0} \to Y$ be defined by $g(x_i, y_i) = f(x_i + \epsilon, y_i)$. Define $\Phi : \Omega_{r_0} \to X_{r_0}$ by setting $\Phi(x_i, y_i) = (x_i, |y_i|)$ and let $F, G : \Omega_{r_0} \to (Y, d)$ be defined by $F = f \circ \Phi$, and $G = g \circ \Phi$ respectively.

For $\eta \in C_c^\infty(D_{r_0})$, define $F_\eta, F_{1-\eta} : \Omega_{r_0} \to (Y, d)$ by setting

$$F_\eta(x_i, y_i) = (1 - \eta(x_i, y_i))F(x_i, y_i) + \eta(x_i, y_i)G(x_i, y_i)$$

and

$$F_{1-\eta}(x_i, y_i) = \eta(x_i, y_i)F(x_i, y_i) + (1 - \eta(x_i, y_i))G(x_i, y_i).$$

where $(1-t)P + tQ$ denotes the point which is fraction t of the way along the geodesic from P to Q in Y. These maps are well-defined with respect to the equivalence relation $\approx$; for example,

$$\begin{aligned}
F_\eta(x_i, 0) &= (1 - \eta(x_i, 0))F(x_i, 0) + \eta(x_i, 0)G(x_i, 0) \\
&= (1 - \eta(x_i, 0))f \circ \Phi(x_i, 0) + \eta(x_i, 0)g \circ \Phi(x_i, 0) \\
&= (1 - \eta(x_i, 0))f(x, 0) + \eta(x_i, 0)g(x, 0) \\
&= (1 - \eta(x_j, 0))f \circ \Phi(x_j, 0) + \eta(x_j, 0)g \circ \Phi(x_j, 0) \\
&= (1 - \eta(x_j, 0))F(x_j, 0) + \eta(x_j, 0)G(x_j, 0)
\end{aligned}$$

$$= \quad F_\eta(x_j, 0).$$

By Lemma 2.4.1 of [**KS1**], we see that the restriction of $F_\eta, F_{1-\eta}$ to $D_{r_0,i}$ is in $W^{1,2}$ so $F_\eta, F_{1-\eta} \in W^{1,2}(\Omega_{r_0}, Y)$ and by Lemma 2.4.2 of [**KS1**] (again applying the lemma to the restriction of $F_\eta, F_{1-\eta}$ to $D_{r_0,i}$), we obtain the inequality

$$\begin{aligned} &\int_{\Omega_{r_0}} |\nabla F_\eta|^2 + \int_{\Omega_{r_0}} |\nabla F_{1-\eta}|^2 \\ &\quad \leq \int_{\Omega_{r_0}} |\nabla F|^2 + \int_{\Omega_{r_0}} |\nabla G|^2 - 2\int_{\Omega_{r_0}} \nabla\eta \cdot \nabla d^2(F,G) + \int_{\Omega_{r_0}} Q(\eta, \nabla\eta). \end{aligned}$$

Let $f_\eta, f_{1-\eta} : X_{r_0} \to Y$ be defined by $f_\eta = F_\eta \circ \Phi^{-1}$, $f_{1-\eta} = F_{1-\eta} \circ \Phi^{-1}$ (where we let $\Phi^{-1}(x_i, y_i) = (x_i, y_i)$) respectively. Since Φ is a local isometry almost everywhere,

$$\int_{\Omega_{r_0}} |\nabla F|^2 = 2\int_{X_{r_0}} |\nabla f|^2 \leq 2\int_{X_{r_0}} |\nabla f_\eta|^2 = \int_{\Omega_{r_0}} |\nabla F_\eta|^2$$

and

$$\int_{\Omega_{r_0}} |\nabla G|^2 = 2\int_{X_{r_0}} |\nabla g|^2 \leq 2\int_{X_{r_0}} |\nabla f_{1-\eta}|^2 = \int_{\Omega_{r_0}} |\nabla F_{1-\eta}|^2.$$

Therefore,

$$-2\int_{\Omega_{r_0}} \nabla\eta \cdot \nabla d^2(F,G) + \int_{\Omega_{r_0}} Q(\eta, \nabla\eta) \geq 0$$

and replacing η by $t\eta$, dividing by t and letting $t \to 0$, we get

$$-\int_{D_{r_0}} \nabla\eta \cdot \nabla\delta \geq 0$$

for any $\eta \in C_c^\infty(D_{r_0})$ and where

$$\delta(x,y) = \sum_{i=1}^n d^2(F(x_i,y_i), G(x_i,y_i)).$$

By the mean value inequality for subharmonic functions,

$$\delta(x,y) \leq \frac{1}{\pi\rho^2}\int_{D_\rho(x,y)} \delta$$

for any $\rho \leq r_0 - \sqrt{x^2+y^2}$ where $D_\rho(x,y)$ is the disk of radius ρ centered at (x,y).

Now suppose $r < r_0$. Let $(x_i, y_i) \in X_r$ and $\rho = r_0 - r$. (As before, D_{r_0} is the disk of radius r_0 centered at the origin.) Then

$$\begin{aligned} d^2(f(x_i,y_i), g(x_i,y_i)) &= d^2(F(x_i,y_i), G(x_i,y_i)) \\ &\leq \delta(x,y) \\ &\leq \frac{1}{\pi\rho^2}\int_{D_\rho(x,y)} \delta \\ &= \frac{1}{\pi\rho^2}\int_{D_{r_0}} \delta \\ &= \frac{1}{\pi\rho^2}\int_{\Omega_{r_0}} d^2(F,G) \\ &= \frac{2}{\pi\rho^2}\int_{X_{r_0}} d^2(f,g). \end{aligned}$$

Thus, for $(\bar{x}_i, \bar{y}_i) \in X_r$ and $\rho = \frac{1-r}{2}$ and $r_0 = r + \rho$.

$$
\begin{aligned}
\left|\frac{\partial f}{\partial x}\right|^2 (\bar{x}_i, \bar{y}_i) &= \lim_{\epsilon \to 0} \frac{d^2(f(\bar{x}_i, \bar{y}_i), f(\bar{x}_i + \epsilon, \bar{y}_i))}{\epsilon^2} \\
&\leq \lim_{\epsilon \to 0} \sum_{j=1}^{n} \frac{d^2(f(\bar{x}_j, \bar{y}_j), f(\bar{x}_j + \epsilon, \bar{y}_j))}{\epsilon^2} \\
&\leq \frac{2}{\pi \rho^2} \sum_{j=1}^{n} \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_{X_{r_0}} d^2(f(x,y), f^j(x+\epsilon, y)) dxdy \\
&\leq \frac{2}{\pi \rho^2} \int_{X_{r_0}} \left|\frac{\partial f}{\partial x}\right|^2 dxdy \\
&\leq \frac{2}{\pi \rho^2} E^f.
\end{aligned}
$$

□

Using Lemma 5, we now show that a harmonic map $f : X_1 \to (Y, d)$ is β-Hölder continuous in X_r for any $\beta < 1$ and $r < 1$.

LEMMA 6. *Let $f : X_1 \to Y$ be a harmonic map. For any $\beta < 1$ and $r < 1$, there exists a constant c_β so that $d(f(z_1), f(z_2)) \leq c_\beta |z^1 - z^2|^\beta$ for $z^1, z^2 \in X_r$.*

PROOF. By Lemma 5, f restricted to Γ_r is Lipschitz continuous. Thus, for each i, we can construct a $W^{1,2}$ map $\phi : D_i^+ \to Y$ so that $f = \phi$ on ∂D_i^+ and ϕ is Lipschitz continuous in a neighborhood of Γ_r as the following shows. Let D be the unit disk and $\psi : D \to D_i^+$ a bi-Lipschitz map. With (r, θ) the polar coordinates of D, define $\Phi : D \to Y$ by setting $\Phi(r, \theta)$ to be the point on the geodesic from $f(0)$ to $f \circ \psi(1, \theta)$ that is a distance r from $f(0)$. The NPC condition on Y guarantees that Φ is Lipschitz continuous near $\psi^{-1}(\Gamma_r)$ and hence $\phi = \Phi \circ \psi^{-1}$ is also Lipschitz near Γ_r and by construction $\phi = f$ on ∂D_i^+. Now Lemma 6 follows from Theorem 3. □

5. The monotonicity formula

The following monotonicity property will be crucial in our regularity proof. For $-1 < x < 1$ and $0 < r < 1 - x$, let $B_r(x, 0)$ be a ball of radius r at $(x, 0) \in X_1$ and set

$$E_x^f(r) = \int_{B_r(x,0)} |\nabla f|^2 d\mu$$

and

$$I_x^f(r) = \int_{\partial B_r(x,0)} d^2(f, f(x,0)) ds.$$

LEMMA 7. *Let $f : X_1 \to Y$ be a harmonic map. Then*

$$r \mapsto Ord(x, r) = \frac{r E_x^f(r)}{I_x^f(r)}$$

is a non-decreasing function for $0 \leq r \leq 1 - x$.

PROOF. For simplicity, we assume $x = 0$ and set $E(r) = E_0^f(r)$ and $I(r) = I_0^f(r)$. Let $\eta : X \to \mathbf{R}_+ \cup \{0\}$ be a continuous function which is smooth on each face of X_1. For η with $spt(\eta) \subset X_r$ and t sufficiently small, we define $F_t : X \to X$ as

$$F_t(x_i, y_i) = ((1 + t\eta(x_i, y_i))x_i, (1 + t\eta(x_i, y_i))y_i).$$

With that, we can now follow the usual calculation to prove Lemma 7. In other words, the standard computation (see [**GS**], Section 2 for example) done on each face of X_1 gives

$$E'(r) = 2 \int_{\partial X_r} \left| \frac{\partial f}{\partial r} \right|^2 ds \tag{2}$$

for a.e. $0 \leq r \leq 1$. Again, for a.e. $0 \leq r \leq 1$, standard computation on each face of X gives,

$$I'(r) = \int_{\partial X_r} \frac{\partial}{\partial r} d^2(f, f(p_0)) ds + \frac{I(r)}{r}.$$

Using the inequality $|\frac{\partial}{\partial r} d(f, f(p_0))| \leq |\frac{\partial f}{\partial r}|$ and the Schwarz inequality, the above two equations imply that

$$\frac{d}{dr} \log \left(\frac{rE(r)}{I(r)} \right) = \frac{1}{r} + \frac{E'(r)}{E(r)} - \frac{I'(r)}{I(r)} \geq 0$$

for a.e. $0 \leq r \leq 1$. $\square$

LEMMA 8. *Let $f : X_1 \to (Y, d)$ be a harmonic map so that*

$$\alpha = \lim_{\sigma \to 0} \frac{\sigma E_0^f}{I_0^f(\sigma)}.$$

Then

$$\frac{2\alpha + 1}{\sigma} \leq \frac{\frac{dI_0^f}{ds}(\sigma)}{I_0^f(\sigma)} \tag{3}$$

and hence

$$\frac{d}{d\sigma} \left(\frac{I_0^f(\sigma)}{\sigma^{2\alpha+1}} \right) \geq 0.$$

PROOF. The argument of [**GS**], Section 2 (see also [**EF**]) implies

$$2 \int_{X_1} |\nabla f|^2 \eta dx dy \leq - \int_{X_1} \nabla d^2(f, f(0)) \cdot \nabla \eta dx dy \tag{4}$$

for any $W^{1,2}$ function η with compact support in X_1. Choosing η to approximate the characteristic function of X_r,

$$2E_0^f(\sigma) \leq \int_{\partial B_\sigma(p_0)} \frac{\partial}{\partial r} d^2(f, f(v)) ds = \frac{d}{d\sigma} I_0^f(\sigma) - \frac{1}{\sigma} I_0^f(\sigma).$$

The monotonicity property of $\sigma \mapsto Ord(0, \sigma)$ implies

$$\frac{\alpha I(\sigma)}{\sigma} \leq E(\sigma)$$

and by combining the two inequalities, we get

$$2\alpha I_0^f(\sigma) \leq \sigma \frac{d}{d\sigma} I_0^f(\sigma) - I_0^f(\sigma)$$

which lead to the desired inequality. $\square$

We prove the following regularity result based on the order α of a harmonic map.

THEOREM 9. *Let $f : X_1 \to (Y,d)$ be a harmonic map and*

$$\alpha = \lim_{\sigma \to 0} \frac{\sigma E_0^f(\sigma)}{I_0^f(\sigma)}.$$

then f satisfies

$$f(z) \leq C|z|^\alpha$$

for all $z = (x_i, y_i) \in X_{\frac{1}{2}}$ where C depends only on α and E^f.

PROOF. For simplicity, we let $E(\sigma) = E_0^f(\sigma)$ and $I(\sigma) = I_0^f(\sigma)$. From Lemma 8, we have

$$\frac{2\alpha + 1}{\sigma} \leq \frac{I'(\sigma)}{I(\sigma)}$$

and integrating this differential inequality from $\sigma \in (0,1)$ to 1, we obtain

$$\frac{I(\sigma)}{\sigma} \leq \sigma^{2\alpha} I(1).$$

The monotonicity property of $\sigma \mapsto Ord(0,\sigma)$ implies

$$I(1) \leq \frac{E(1)}{\alpha},$$

and hence

$$\frac{I(\sigma)}{\sigma} \leq \frac{E(1)}{\alpha}\sigma^{2\alpha}.$$

Define $\delta : D \to \mathbf{R}$ be setting

$$\delta(x,y) = \begin{cases} \sum_{i=1}^N d^2(f(x^i, y^i), 0) & y_i \geq 0 \\ \sum_{i=1}^N d^2(f(x^i, |y^i|), 0) & y_i < 0. \end{cases}$$

Using the argument of the proof of Lemma 5 with $g(x_i, y_i) = 0$ instead of $g(x_i, y_i) = f(x_i + \epsilon, y_i)$, we can show that δ is a subharmonic function and the mean value inequality implies

$$\begin{aligned} \sup_{(x_i,y_i) \in X_{\frac{\sigma}{2}}} d^2(f(x_i,y_i), f(0)) &\leq \sup_{z \in D_{\frac{\sigma}{2}}} \delta(x,y) \\ &\leq \frac{4}{\pi\sigma} \int_{\partial D_\sigma} \delta ds \\ &\leq \frac{4I(\sigma)}{\pi\sigma} \\ &\leq C\sigma^{2\alpha} \end{aligned}$$

where $C = \frac{4E(1)}{\alpha\pi}$. □

Therefore, the Lipschitz continuity of $f : X_1 \to Y$ if we show that $\alpha \geq 1$. Hence, we will now assume that $\alpha < 1$ and show that this leads to a contradiction.

6. The tangent map

LEMMA 10. *Let $f : X_1 \to Y$ be a harmonic map. Fix $0 < \beta < 1$. For any $x \in [-\frac{1}{2}, \frac{1}{2}]$ and $\sigma < \frac{1}{4}$, there exists a constant c_1 so that $E^f_x(\sigma) \leq c_1 \sigma^{2\beta}$.*

PROOF. Let

$$k_1 = \max_{x \in [-\frac{1}{2}, \frac{1}{2}]} \frac{E_x(\frac{1}{4})}{4I_x(\frac{1}{4})}.$$

By Lemma 6, there exists k_2 so that $d(f(z^1), f(z^2)) \leq k_2 |z^1 - z^2|^\beta$ for z^1, z^2 in a neighborhood of $\Gamma_{\frac{3}{4}}$. Thus, $I_x(\sigma) \leq k_2 \sigma^{2\beta+1}$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and $\sigma < \frac{1}{4}$. The monotonicity property of Lemma 7 shows that

$$E_x(\sigma) \leq \frac{E_x(\frac{1}{4})}{4I_x(\frac{1}{4})} \frac{I_x(\sigma)}{\sigma} = c_1 \sigma^{2\beta}$$

where $c_1 = k_1 k_2$. □

An important consequence of the monotonicity property of Lemma 7 is the existence of a tangent map.

LEMMA 11. *Let $f : X_1 \to (Y, d)$ be a harmonic map,*

$$\mu^f(\lambda) = \sqrt{\frac{\lambda}{I^f_0(\lambda)}}$$

and $d_\lambda(\cdot, \cdot) = \mu^f(\lambda) d(\cdot, \cdot)$. Define $f_\lambda : X_1 \to (Y, d_\lambda)$ by setting

$$f_\lambda(x_i, y_i) = f(\lambda x_i, \lambda y_i).$$

Assume that

$$\alpha = \lim_{\sigma \to 0} \frac{\sigma E^f_0(\sigma)}{I^f_0(\sigma)} < 1.$$

There exists $\lambda_k \to 0$, an NPC space (Y_*, d_*) and a harmonic map $f_* : X_1 \to (Y_*, d_*)$ so that $d_{\lambda_k}(f_{\lambda_k}(\cdot), f_{\lambda_k}(\cdot))$ converges uniformly to $d_*(f_*(\cdot), f_*(\cdot))$ and $E^{f_{\lambda_k}}(r) \to E^{f_*}(r)$ for $0 < r \leq 1$.

PROOF. Let $\lambda_k \to 0$. Since

$$E^{f_{\lambda_k}}_0(1) = (\mu^f(\lambda_k))^2 E^f_0(\lambda_k) = \frac{\lambda_k E^f_0(\lambda_k)}{I_0(\lambda_k)} \leq \frac{E^f_0(1)}{I^f_0(1)},$$

the energy of f_{λ_k} is uniformly bounded. The local Hölder constant and exponent of a harmonic map from X_1 to Y can be shown to be dependent only on the energy of the map and the distance to ∂X_1. (For example, we can use the proof of [**Ch**]. We note that although the proof of [**Ch**] does not apply to a general metric as is considered in [**EF**], it works in our case when the domain is flat. In [**Ch**], we clearly see the dependence of the Hölder constant and exponent only on the total energy and the distance to the boundary.) Therefore, f_{λ_k} satisfies the uniform modulus of continuity control needed to apply Proposition 3.7 of [**KS2**].

For $\lambda_k \to 0$ and any $\tau < 1$, we see that there exists a subsequence of f_{λ_k} which converges locally uniformly in the pullback sense to a harmonic map by applying the argument of [**KS2**] Proposition 3.7. Pick a sequence $\tau_n \to 1$ and by a diagonalization procedure, we can pick a subsequence f_{λ_k} (which we again denote by $\{f_{\lambda_k}\}$ by an abuse of notation) which converges locally uniformly in the pull back sense to a harmonic map $f_*: X_1 \to (Y, d_*)$. In other words, $d_{\lambda_k}(f_{\lambda_k}(\cdot), f_{\lambda_k}(\cdot))$ converges uniformly to $d_*(f_*(\cdot), f_*(\cdot))$.

We first show that f_* is non-constant. Let $E^k(r) = E_0^{f_{\lambda_k}}(r)$, $I^k(r) = I_0^{f_{\lambda_k}}(r)$, $E^*(r) = E_0^{f_*}(r)$, $I^*(r) = I_0^{f_*}(r)$ for simplicity. Repeating the computation of [**GS**], proof of Proposition 3.3,

$$I^k(r_0) - I^k(\theta) \le \epsilon E^k(r_0) + \left(\frac{1}{\epsilon} + \frac{1}{\theta}\right) \int_\theta^{r_0} I^k(r)dr$$

for $0 < \theta < r < 1$ and any $\epsilon > 0$. By Lemma 7,

$$\frac{r_0 E^k(r_0)}{I^k(r_0)} \le \frac{E^k(1)}{I^k(1)} = C$$

and hence

$$I^k(r_0) - I^k(\theta) \le \frac{\epsilon C I^k(r_0)}{r_0} + \left(\frac{1}{\epsilon} + \frac{1}{\theta}\right) \int_\theta^{r_0} I^k(r)dr.$$

For any $\theta \in [\frac{1}{2}, 1)$, pick $r_0 \in (\theta, 1]$. Then $r_0 \ge \theta \ge \frac{1}{2}$ and by choosing $\epsilon = \frac{1}{4C}$,

$$\begin{aligned}
\frac{1}{2} I^k(r_0) - I^k(\theta) &\le (4C + \frac{1}{\theta}) \int_\theta^{r_0} I^k(r)dr \\
&\le (4C+2) \int_\theta^1 I^k(r)dr \\
&\le (4C+2)(1-\theta) \sup_{r \in [\theta, \sigma]} I^k(r).
\end{aligned}$$

Since r_0 is an arbitrary point in $(\theta, 1]$, we have

$$\frac{1}{2} \sup_{r \in [\theta, 1]} I^k(r) - I^k(\theta) \le (4C+2)(1-\theta) \sup_{r \in [\theta, \sigma]} I^k(r).$$

Now choose θ sufficiently close to 1 so that $(4C+2)(1-\theta) \le \frac{1}{6}$. Then

$$I^k(\theta) \ge \frac{1}{3} \sup_{r \in [\theta, 1]} I^k(r) \ge \frac{1}{3} I^k(1) = \frac{1}{3}.$$

By the uniform convergence of $d_{\lambda_k}(f_{\lambda_k}(\cdot), f_{\lambda_k}(\cdot))$ to $d_*(f_*(\cdot), f_*(\cdot))$, we then have

$$I^*(\theta) \ge \frac{1}{3}$$

and this shows that f_* is non-constant.

Finally, we show that $E^k(r) \to E^*(r)$ for $r < 1$. By Theorem 3.11 of [**KS2**], it will be enough to show that

$$E^k_{\epsilon-\text{strip}} := \int_{\Gamma_{1,\epsilon}} |\nabla f_k|^2 dxdy < C_1 \epsilon^\gamma$$

for $\gamma > 0$. Choose β so that $\max\{\alpha, \frac{1}{2}\} < \beta < 1$. Let $\epsilon > 0$ be given. For $\lambda < \frac{1}{4}$, let

$$E^f_{\epsilon\lambda-\text{strip}} = \int_{\Gamma_{\lambda,\lambda\epsilon}} |\nabla f|^2 dxdy$$

Note that $\Gamma_{\lambda,\lambda\epsilon}$ can be covered by $\frac{10}{\epsilon}$ number of balls of radius $2\lambda\epsilon$ centered at points on $\Gamma_\lambda \subset \Gamma_{\frac{1}{2}}$. Therefore, by Lemma 10

$$E^f_{\epsilon\lambda-\text{strip}} \le \frac{10}{\epsilon} c_1 (2\lambda\epsilon)^{2\beta}.$$

Since

$$\alpha = \lim_{\sigma\to 0} \frac{\sigma E_0^f(\sigma)}{I_0^f(\sigma)}.$$

we know

$$\alpha I_0^f(\sigma) \le \sigma E_0^f(\sigma)$$

by Lemma 7.

Now letting $C_2 = \lim_{\sigma\to 0} \frac{I_0^f(\sigma)}{\sigma^{2\alpha+1}}$, we see that $C_2 \le \frac{I_0^f(\lambda_k)}{\lambda_k^{2\alpha+1}}$ by Lemma 8 or equivalently

$$(\mu^f(\lambda_k))^2 = \frac{\lambda_k}{I_0^f(\lambda_k)} \le C_2\lambda_k^{-2\alpha}.$$

By the definition of $f_{\lambda_k} : X_1 \to (Y, d_{\lambda_k})$, we have

$$E^k_{\epsilon-\text{strip}} = (\mu^f(\lambda_k))^2 E^f_{\epsilon\lambda_k-\text{strip}} \le C_2\lambda_k^{-2\alpha}\frac{10}{\epsilon}c_1(2\lambda_k\epsilon)^{2\beta} = 10C_2c_1\lambda_k^{2\beta-2\alpha}\epsilon^{2\beta-1}$$

We have chosen β so that $2\beta - 2\alpha > 0$ and $2\beta - 1 > 0$, so letting $C_1 = 10C_2c_1$ and $\gamma = 2\beta - 1$, we have

$$E^k_{\epsilon-\text{strip}} = k_3\epsilon^\gamma.$$

By Theorem 3.11 of [**KS1**], this shows that the energy of f_k on each $D_i^+ \cap X_r$ converges to that of f_* so $E^k(r) \to E^{f_*}(r)$ for all $r < 1$ and hence $E^{f_{\lambda_k}}(1) \to E^{f_*}(1)$ also. □

We call f_* of Lemma 11 a tangent map of f. We note the following property of f_*.

LEMMA 12. *Let $f : X_1 \to (Y, d)$ be a harmonic map so that*

$$\alpha = \lim_{\sigma\to 0} \frac{\sigma E_0^f(\sigma)}{I_0^f(\sigma)} < 1.$$

Then its tangent map $f_* : X_1 \to (Y, d_*)$ is a homogeneous map of order α; in other words,

$$d_*(f_*(z), f_*(0)) = |z|^\alpha d_*\left(f_*\left(\frac{z}{|z|}\right), f_*(0)\right).$$

PROOF. Since

$$E_0^{f_{\lambda_k}}(r) = (\mu^f(\lambda_k))^2 E_0^f(\lambda_k r) = \frac{\lambda_k}{I_0(\lambda_k)}E_0^f(\lambda_k r)$$

and

$$I_0^{f_{\lambda_k}}(r) = (\mu^f(\lambda_k))^2\frac{I_0^f(\lambda_k r)}{\lambda_k} = \frac{\lambda_k}{I_0^f(\lambda_k)}\frac{I_0^f(\lambda_k r)}{\lambda_k} = \frac{I_0^f(\lambda_k)}{I_0^f(\lambda_k r)},$$

we have

$$\frac{rE_0^{f_*}(r)}{I_0^{f_*}(r)} = \lim_{k\to\infty}\frac{rE_0^{f_{\lambda_k}}(r)}{I_0^{f_{\lambda_k}}(r)} = \lim_{k\to\infty}\frac{\lambda_k r E_0^f(\lambda_k r)}{I_0^f(\lambda_k r)} = \alpha.$$

Therefore, by proof of Lemma 3.2 of [**GS**], f_* is a homogeneous map of order α. □

7. The Lipschitz continuity

We are now ready to prove:

LEMMA 13. *The harmonic map $f : X_1 \to Y$ is Lipschitz continuous at* 0 *with the Lipschitz constant dependent only on E^f.*

PROOF. By Theorem 9, it is sufficient to show that $\alpha \geq 1$. Suppose $\alpha < 1$. Let $f_* : X_1 \to Y_*$ be a tangent map of f. By Lemma 5, f_* satisfies

$$d_*^2(f_*(x_i, y_i), f_*(0, y_i)) \leq L^2 x_i^2.$$

Let $f_{**} : X_1 \to Y_{**}$ be a tangent map of f_* so that for $\lambda_k \to 0$, $f_{*,\lambda_k} : X_1 \to (Y_*, d_{*,\lambda_k})$ converges in the sense of Lemma 11 to f_{**}. By Lemma 12, both f_* and f_{**} are homogeneous maps of order α. Then

$$\begin{aligned} & d_{**}(f_{**}(x_i, y_i), f_{**}(0, y_i)) \\ &= \lim_{k\to\infty} d_{*,\lambda_k}(f_{*,\lambda_k}(x_i, y_i), f_{*,\lambda_k}(0, y_i)) \\ &= \lim_{k\to\infty} \mu^{f_*}(\lambda_k) d_*(f_*(\lambda_k x_i, \lambda_k y_i), f_*(0, \lambda_k y_i)) \\ &= \lim_{k\to\infty} \mu^{f_*}(\lambda_k) L \lambda_k x_i \end{aligned}$$

Furthermore,

$$I^{f_*}(\lambda_k) = \int_{\partial X_{\lambda_k}} d^2(f_*, f_*(0)) ds = \lambda_k^{2\alpha+1} \int_{\partial X_1} d_*^2(f_*, f_*(0)) ds = \lambda_k^{2\alpha+1} I^{f_*}(1)$$

so we have

$$\begin{aligned} \mu^{f_*}(\lambda_k) &= \sqrt{\frac{\lambda_k}{I^{f_*}(\lambda_k)}} \\ &= \lambda_k^{-\alpha} (I^{f_*}(1))^{-\frac{1}{2}}. \end{aligned}$$

Therefore,

$$d_{**}(f_{**}(x_i, y_i), f_{**}(0, y_i)) = \lim_{k\to\infty} L \lambda_k^{1-\alpha} x_i (I^{f_*}(1))^{-\frac{1}{2}} = 0.$$

This, in particular, shows that $t \mapsto f_{**}(t_i, y_i)$ is a geodesic curve. Therefore, $\alpha = 1$ and this is a contradiction. Therefore, $\alpha \geq 1$. It follows from the proof of Theorem 9 that the Lipschitz constant depends only on E^f. □

Furthermore, it is now straightforward to show that the Lipschitz constant of a harmonic map $f : X \to Y$ at point $(x, 0)$ for $|x| < \frac{1}{2}$ depends only on E^f. We now show:

LEMMA 14. *The local Lipschitz constant of $f : X_1 \to Y$ at a point $z_0 = (x_i, y_i) \in X_{\frac{1}{2}}$ with $y_i \neq 0$ depends only on E^f.*

PROOF. Let $r_0 = y_i$ and

$$E^f_{(x_i,y_i)}(r) = \int_{B_r(x_i,y_i)} |\nabla f|^2 d\mu$$

and

$$I^f_{(x_i,y_i)}(r) = \int_{\partial B_r(x_i,_i)} d^2(f, f(x_i, y_i)) ds.$$

Then

$$r \mapsto \frac{rE^f_{(x_i,y_i)}(r)}{I^f_{(x_i,y_i)}(r)}$$

is monotone for $r < r_0$ since $B_r(x_i, y_i)$ is contained in a 2-simplex for $r < r_0$. As in Lemma **??**, we can deduce that

$$\frac{I^f_{(x_i,y_i)}(r)}{r} \leq \frac{I^f_{(x_i,y_i)}(r_0)}{r_0^{2\alpha+1}} r^{2\alpha} \leq \frac{E^f_{(x_i,y_i)}(r_0)}{\alpha r_0^{2\alpha}} r^{2\alpha}$$

where $\alpha = \alpha(x_i, y_i) = \lim_{\sigma \to 0} \frac{\sigma E^f_{(x_i,y_i)}(\sigma)}{I^f_{(x_i,y_i)}(\sigma)}$. Therefore,

$$\begin{aligned} \sup_{z \in B_{\frac{r}{2}}(z_0)} d^2(f(z), f(z_0)) &\leq \frac{4I^f_{(x_i,y_i)}(r)}{\pi r} \\ &\leq \frac{4E^f_{(x_i,y_i)}(r_0)}{\pi \alpha r_0^{2\alpha}} r^{2\alpha} \end{aligned}$$

We know that $\alpha \geq 1$ since $(x_i, y_i) \mapsto \alpha(x_i, y_i)$ is upper semicontinuous since it is a decreasing limit of continuous functions. If $\alpha > 1$, then

$$0 \leq \lim_{z \mapsto z_0} \frac{d^2(f(z), f(z_0))}{|z - z_0|^2} \leq \frac{4E^f_{(x_i,y_i)}(r_0)}{\pi \alpha r_0^{2\alpha}} \lim_{r \to 0} r^{2\alpha - 2} = 0$$

If $\alpha = 1$, then

$$\lim_{z \mapsto z_0} \frac{d^2(f(z), f(z_0))}{|z - z_0|^2} \leq \frac{4E^f_{(x_i,y_i)}(r_0)}{\pi r_0^2} \leq \frac{16E^f_{(x_i,0)}(2r_0)}{\pi (2r_0)^2} \leq \frac{16E^f}{\pi}$$

This shows that the energy density function $|\nabla f|^2$ is uniformly bounded by a constant dependent on energy and proves Lemma 14. □

Lemma 13 and Lemma 14 combines to prove Theorem 1 for two dimensional domain. As mentioned previously, it is straightforward to generalize the arguments here to a higher dimensional domain.

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