Surface groups acting on CAT(−1) spaces

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Abstract. Harmonic map theory is used to show that a convex cocompact surface group action on a CAT(−1) metric space fixes a convex copy of the hyperbolic plane if and only if the Hausdorff dimension of the limit set of the action is equal to 1. This provides another proof of a result of Bonk and Kleiner.

1 Introduction

The relationship between dynamical properties of discrete group actions on metric spaces and rigidity theorems has a rich history: a prototypical result due to Bowen [Bow] states that the Hausdorff dimension of the limit set of a quasi-Fuchsian surface group Γ acting on hyperbolic 3-space is equal to 1 if and only if Γ restricts to an isometric action on hyperbolic 2-space (Γ is Fuchsian).

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A generalization of this result to surface group actions on CAT(-1) metric spaces was originally conjectured by Bourdon [Bou]. It was later verified by Bonk and Kleiner [BK] where they prove a more general statement about quasi-convex actions.

In the special case when the space is a smooth Riemannian manifold with pinched negative sectional curvature, a different proof of this result was found by one of the authors [San] utilizing equivariant harmonic maps and an inequality which compared curvature quantities of the harmonic map with the Hausdorff dimension of the limit set. Here, we extend these techniques to the general CAT(-1) setting using the harmonic map theory which has been developed by Koraveer-Schoen [KS1,KS2] and Mese [Mes] to give a new proof of the original conjecture of Bourdon.

The main theorem of this paper is the following:

Theorem 1 *Given a convex cocompact action $\rho : \pi_1(S) \to Isom(X)$ on a CAT(-1) metric space X by the fundamental group of a closed, connected oriented surface S with genus > 1 such that $\dim_{\mathcal{H}}(\Lambda) = 1$, there exists a hyperbolic metric h on the universal cover $\tilde{S}$ of S such that the unique ρ-equivariant harmonic map $u : \mathbb{H}^2 = (\tilde{S}, h) \to X$ is totally geodesic and an isometric embedding. Here, Λ is the limit set of this action and the Hausdorff dimension is computed using any of the Gromov metrics on $\partial_\infty X$.*

In the Riemannian setting, the above theorem has been proved by one of the authors [San] and also by Deroin-Tholozan [DT]. In the CAT(-1) setting, the work of Mese [Mes] establishes the existence of a metric (of low regularity) on S for which the associated equivariant harmonic map is conformal. Moreover, where it makes sense, this metric is the pullback metric of the associated harmonic map. The main challenge overcome in this paper is a comparison result (see Lemma 13) which allows us to trade this irregular metric for a smooth metric in the same conformal class to which the arguments of [San] can be directly applied.

Theorem 1 immediately yields a new proof of the conjecture of Bourdon [Bou]:

Corollary 2 *A convex cocompact isometric action $\rho : \pi_1(S) \to Isom(X)$ on a CAT(-1) space X by the fundamental group of a closed, connected oriented surface S with genus > 1 fixes a convex copy of $\mathbb{H}^2$ if and only if $\dim_{\mathcal{H}}(\Lambda_\Gamma) = 1$.*

The paper is organized as follows. A brief preliminary section settles notation and states the fundamental existence result for equivariant harmonic maps of surfaces to CAT(-1) metric spaces. Section 3 establishes the primary technical result eluded to in the introduction which replaces a conformal metric of low regularity with a comparable smooth Riemannian metric. Section 4 recapitulates the basic arguments of [San] establishing an inequality relating the curvature of an equivariant conformal harmonic map to the Hausdorff dimension of the limit set. Finally, section 5 completes the proof of Theorem 1 from which Collorary 2 immediately follows.

2 Preliminaries

Throughout this paper, S denotes a closed, connected oriented surface of genus > 1, $\tilde{S} \simeq \mathbb{H}^2$ its universal cover, and $\mathcal{T}$ is the Teichmüller space of S. Furthermore, X denotes a CAT(-1) space, $\mathrm{Isom}(X)$ is the group of isometries of X and $\rho : \pi_1(S) \to \mathrm{Isom}(X)$ is a representation. We will identify an element of $\pi_1(S)$ as a deck transformation of $\tilde{S}$ and an element $h \in \mathcal{T}$ as an equivariant hyperbolic metric on $\tilde{S}$ (via uniformization).

Definition 3 Given a representation $\rho : \pi_1(S) \to \mathrm{Isom}(X)$, a map $\tilde{u} : \tilde{S} \to X$ is said to be *ρ-equivariant* if

$$\tilde{u}(\gamma x) = \rho(\gamma)\tilde{u}(x), \quad \forall x \in \tilde{S}, \ \gamma \in \pi_1(S).$$

Definition 4 A discrete subgroup Γ of $\mathrm{Isom}(X)$ is said to be *convex cocompact* if there exists a geodesically convex, Γ-invariant subset of X upon which Γ acts cocompactly. A representation $\rho : \pi_1(S) \to \mathrm{Isom}(X)$ is said to be *convex cocompact* if $\Gamma = \rho(\pi_1(S))$ is convex cocompact.

Given a convex cocompact representation as above, a ρ-equivariant map $\tilde{u} : \tilde{S} \to X$ descends to a map

$$u : S \to X/\Gamma. \tag{1}$$

Definition 5 Given a hyperbolic metric $h \in \mathcal{T}$, the *energy* of a ρ-equivariant map $\tilde{u} : (\tilde{S}, h) = \mathbb{H}^2 \to X$ is

$$E_h^{\tilde{u}} = \int_F |\nabla \tilde{u}|^2 dh,$$

where $F \subset \mathbb{H}^2$ is a fundamental of the action of $\pi_1(S)$, $|\nabla \tilde{u}|^2$ is the energy density function of $\tilde{u}$ as defined in [KS1], and dh is the volume form associated with the metric h. Furthermore, we will denote

$$E_h^{\tilde{u}}[U] = \int_U |\nabla \tilde{u}|^2 dh$$

for any measurable set $U \subset \mathbb{H}$.

Definition 6 Given a hyperbolic metric $h \in \mathcal{T}$, a finite energy map $\tilde{u} : (\tilde{S}, h) = \mathbb{H}^2 \to X$ is said to be *harmonic* if $E_h^{\tilde{u}}[\Omega] \leq E_h^{\tilde{v}}[\Omega]$ for any bounded Lipschitz domain $\Omega \subset \tilde{S}$ and any finite energy map $\tilde{v} : \Omega \to X$ with same boundary values as $\tilde{u}$.

Theorem 7 (cf. [KS1], [KS2]) *Given a convex cocompact representation $\rho : \pi_1(S) \to X$ and a hyperbolic metric $h \in \mathcal{T}$, there exists a unique ρ-equivariant harmonic map $\tilde{u} : (\tilde{S}, h) = \mathbb{H}^2 \to X$. Furthermore, $\tilde{u}$ is locally Lipschitz continuous.*

3 Conformal Harmonic Maps

Definition 8 Given a hyperbolic metric $h \in \mathcal{T}$ and a finite energy map $\tilde{u} : (\tilde{S}, h) = \mathbb{H}^2 \to X$, let π be the pullback inner product structure defined in [KS1] Theorem 2.3.2. If $z = x + iy$ is a local conformal coordinate on $(\tilde{S}, h)$ and $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ are the coordinate vector fields, we obtain the locally integrable functions

$$\left|\frac{\partial \tilde{u}}{\partial x}\right|^2 := \pi(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}), \quad \left|\frac{\partial \tilde{u}}{\partial y}\right|^2 := \pi(\frac{\partial}{\partial y}, \frac{\partial}{\partial y})$$

and

$$< \frac{\partial \tilde{u}}{\partial x}, \frac{\partial \tilde{u}}{\partial y} > := \pi(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}).$$

Definition 9 Given a hyperbolic metric $h \in \mathcal{T}$, a finite energy map $\tilde{u} : (\tilde{S}, h) = \mathbb{H}^2 \to X$ is said to be *conformal* if

$$\left|\frac{\partial \tilde{u}}{\partial x}\right|^2 = \left|\frac{\partial \tilde{u}}{\partial y}\right|^2 \text{ and } < \frac{\partial \tilde{u}}{\partial x}, \frac{\partial \tilde{u}}{\partial y} >= 0$$

where (x, y) are local conformal coordinates of $(\tilde{S}, h)$. The *local conformal factor* of $\tilde{u}$ is the locally integrable function

$$\lambda = \left| \frac{\partial \tilde{u}}{\partial x} \right|^2.$$

The next result is well known. When the target space is a Riemannian manifold it is due to Schoen-Yau (cf. [SY]), but the argument goes through almost verbatim for the singular targets considered here. See for example [GW].

Proposition 10 *Given a convex cocompact representation $\rho : \pi_1(S) \to Isom(X)$, there exists $h \in \mathcal{T}$ and a ρ-equivariant conformal harmonic map $\tilde{u} : \mathbb{H}^2 = (\tilde{S}, h) \to X$.*

The following theorem is a summary of the results contained in [Mes].

Theorem 11 ([Mes]) *If $h \in \mathcal{T}$ is a hyperbolic metric and $\tilde{u} : (\tilde{S}, h) = \mathbb{H}^2 \to X$ is a conformal harmonic map, then the local conformal factor λ with respect to conformal coordinates of $(x, y) \in U \subset \mathbb{R}^2$ satisfies the following properties:*

(i) $\lambda \in H^1_{loc}(U)$,

(ii) $\log \lambda \in W^{1,1}_{loc}(U)$,

(iii) *λ satisfies the weak differential inequality,*

$$\int_U (\triangle \varphi) \log \lambda \; dxdy \geq 2 \int_U \varphi \lambda \; dxdy, \quad \forall \varphi \in C^\infty_c(U), \varphi \geq 0 \tag{2}$$

where $\triangle$ is the Euclidean Laplacian in coordinates (x, y) and

(iv) *The zero set of λ is of Hausdorff dimension zero; i.e.*

$$\dim_{\mathcal{H}}(\mathcal{D}) = 0 \quad \textit{where} \quad \mathcal{D} = \{z = (x, y) \in U : \lambda(z) = 0\}.$$

In (iv), we let λ be the representative function in the L^1-class defined everywhere by

$$\lambda(z_0) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{D_r(z_0)} \lambda(z) dxdy.$$

Definition 12 Given $\rho : \pi_1(S) \to \text{Isom}(X)$, $h \in \mathcal{T}$ a hyperbolic metric and a ρ-equivariant conformal harmonic map $\tilde{u} : \mathbb{H}^2 = (\tilde{S}, h) \to X$, the *pullback metric* of $\tilde{u}$ is the equivariant (possibly degenerate) two-form G defined locally by

$$G = \lambda(dx^2 + dy^2) \tag{3}$$

where λ is the local conformal factor of $\tilde{u}$.

Lemma 13 *Given $\rho : \pi_1(S) \to Isom(X)$, $h \in \mathcal{T}$ a hyperbolic metric, a ρ-equivariant conformal harmonic map $\tilde{u} : \mathbb{H}^2 = (\tilde{S}, h) \to X$ and G defined by (3), there exists a smooth equivariant conformally equivalent metric G^σ on $\tilde{S}$ satisfying the following properties:*

(i) The Gaussian curvature K^σ of G^σ satisfies $K^\sigma \leq -1$.

(ii) If G^σ is the hyperbolic metric, then G is also the hyperbolic metric.

(iii) The metrics satisfy the inequality

$$G(V,V) \leq G_\sigma(V,V), \quad a.e.\ q \in \tilde{S}, \forall V \in T_q\tilde{S}.$$

Proof. For $\kappa > 0$ to be chosen later, let $g = \kappa^{-1}h$; thus g is a metric on $\tilde{S}$ with constant Gaussian curvature $-\kappa$. Define the ρ-invariant locally integrable function

$$f = \frac{G}{g} : \tilde{S} \to [0, \infty). \tag{4}$$

Let $D = \{(x,y) \in \mathbb{R}^2 : r = \sqrt{x^2+y^2} < 1\}$ and consider the scaled Poincare disk

$$(D, \omega(r)(dx^2 + dy^2))$$

where $D = \{(x,y) \in \mathbb{R}^2 : r = \sqrt{x^2+y^2} < 1\}$ and

$$\omega(r) = \kappa^{-1}\frac{4}{(1-r^2)^2}.$$

In particular,

$$\triangle \log \omega = 2\kappa\omega. \tag{5}$$

Fix $q \in \tilde{S}$ and identify $(\tilde{S}, g)$ to the scaled Poincare disk with q corresponding to the origin. For clarity, we will refer to the coordinates (x,y) as the *Poincare coordinates centered at q*. Fix $\sigma > 0$ and consider a smooth radially symmetric function φ_σ with support contained in $D_\sigma(0) := \{(x,y) : r < \sigma\} \subset D$ and the measure $d\nu_q = \varphi_\sigma(r)dxdy$ on D supported in $D_\sigma(0)$. Via the identification of $\tilde{S}$ to D, we can consider $d\nu_q$ as a measure defined on $\tilde{S}$. Furthermore, by multiplying φ_σ by an appropriate constant if necessary, we can assume that

$$\int_{\tilde{S}} d\nu_q(p) = \int_{D_\sigma(0)} \varphi_\sigma(r)dxdy = 1. \tag{6}$$

If dg is the volume measure associated to the metric g, then we can write dg in the Poincare coordinates (x,y) centered at q as

$$dg = \omega(r)dxdy.$$

Since

$$\frac{d\nu_q}{dg} = \frac{\varphi_\sigma(r)}{\omega(r)}$$

and the hyperbolic distance (i.e. the distance with respect to metric h) of a point (x,y) from $(0,0)$ is dependent only on r, the above quotient is a function dependent only on the distance from q with respect to the metric $g = \kappa^{-1}h$. In other words,

$$d\nu_q(p) = \eta_\sigma(d_g(q,p))dg(p). \tag{7}$$

Furthermore, since g is a scalar multiple of the hyperbolic metric, the function $\eta_\sigma(d_g(q,p))$ has the following property: if we denote the Laplacian in g with respect to variables q and p by $\triangle_g^q$ and $\triangle_g^p$ respectively, then for any $(p_0,q_0) \in \tilde{S} \times \tilde{S}$,

$$\triangle_g^q \eta_\sigma(d_g(q,p))|_{(p,q)=(p_0,q_0)} = \triangle_g^p \eta_\sigma(d_g(q,p))|_{(p,q)=(p_0,q_0)}. \tag{8}$$

Indeed, if I is an isometry of the hyperbolic plane switching the points p_0 and q_0, then

$$\begin{aligned}
\triangle_g^q \eta_\sigma(d_g(q,p))|_{(p,q)=(p_0,q_0)} &= \triangle_g^q \eta_\sigma(d_g(Iq,Ip))|_{(p,q)=(p_0,q_0)} \\
&= \triangle_g^q \eta_\sigma(d_g(q,p))|_{(p,q)=(Ip_0,Iq_0)} \\
&= \triangle_g^q \eta_\sigma(d_g(q,p))|_{(p,q)=(q_0,p_0)} \\
&= \triangle_g^p \eta_\sigma(d_g(p,q))|_{(p,q)=(p_0,q_0)} \\
&= \triangle_g^p \eta_\sigma(d_g(q,p))|_{(p,q)=(p_0,q_0)}.
\end{aligned}$$

We define a smooth function $f^\sigma : \tilde{S} \to [0,\infty)$ by setting

$$f^\sigma(q) = \exp\left(\int_{\tilde{S}} \log f \; d\nu_q\right)$$

and a smooth metric G^σ by setting

$$G^\sigma := f^\sigma g.$$

Since the metric g, the function f and the measure $d\nu_q$ are all ρ-invariant, so is G^σ. Jensen's inequality implies

$$f^\sigma(q) = \exp\left(\int_{\tilde{S}} \log f(p) \; d\nu_q(p)\right) \leq \int_{\tilde{S}} f(p) \; d\nu_q(p). \tag{9}$$

In the Poincare coordinates (x, y) centered at $q \in \tilde{S}$, we will write

$$G = \lambda(dx^2 + dy^2), \quad \lambda = f\omega, \tag{10}$$

$$G^\sigma = \lambda^\sigma(dx^2 + dy^2), \quad \lambda^\sigma = f^\sigma \omega \tag{11}$$

and the function $p \mapsto \eta_\sigma(d_g(p,q))$ as $\eta(r)$. With the above notation, we compute in the Poincare coordinates (x, y) centered at q to obtain

$$\begin{aligned}
&\int_{\tilde{S}} \log f(x,y) \; \triangle\eta(r) dxdy + 2\kappa \\
&= \int_{\tilde{S}} \log \lambda(x,y) \; \triangle\eta(r) dxdy - \int_{\tilde{S}} \log \omega(r) \; \triangle\eta(r) dxdy + 2\kappa \quad \text{(by (10))} \\
&= \int_{\tilde{S}} \log \lambda(x,y) \; \triangle\eta(r) dxdy - 2\kappa \int_{\tilde{S}} \eta(r)\omega(r) dxdy + 2\kappa \quad \text{(by (5))} \\
&= \int_{\tilde{S}} \log \lambda(x,y) \; \triangle\eta(r) dxdy \quad \text{(by (6) and (7))} \\
&\geq 2 \int_{\tilde{S}} \lambda(x,y)\eta(r) dxdy \quad \text{(by Theorem 11)} \\
&= 2 \int_{\tilde{S}} f(x,y)\eta(r)\omega(r) dxdy \quad \text{(by (10))}.
\end{aligned}$$

Since

$$\triangle_g^p \eta_\sigma(d_g(q,p)) dg(p) = \frac{1}{\omega(r)} \triangle\eta_\sigma(r)\omega(r) dxdy = \triangle\eta_\sigma(r) dxdy,$$

we conclude

$$\int_{\tilde S} \log f(p)\ \triangle_g^p \eta_\sigma(d_g(q,p))\ dg(p) + 2\kappa \geq 2\int_{\tilde S} f\eta_\sigma(d_g(q,p))\ dg(p). \tag{12}$$

To obtain the Gaussian curvature of G^σ, we compute

$$\begin{aligned}
&\frac{1}{\omega(0,0)}\triangle \log\lambda^\sigma\Big|_{(0,0)} \\
&= \frac{1}{\omega}\triangle\log f^\sigma\Big|_{(0,0)} + \frac{1}{\omega}\triangle\log\omega\Big|_{(0,0)} \\
&= \triangle_g^q\left(\int_{\tilde S}\log f(p)\ d\nu_q(p)\right) + 2\kappa \\
&= \int_{\tilde S}\log f(p)\ \triangle_g^q\eta_\sigma(d_g(q,p))dg(p) + 2\kappa \\
&= \int_{\tilde S}\log f(p)\ \triangle_g^p\eta_\sigma(d_g(q,p))dg(p) + 2\kappa \qquad \text{(by (8))} \\
&\geq 2\int_{\tilde S} f(p)\eta_\sigma(d_g(q,p))\ dg(p) \qquad \text{(by (12))} \\
&= 2\int_{\tilde S} f(p)\ d\nu_q(p) \qquad \text{(by (7))} \\
&\geq 2f^\sigma(0,0). \qquad \text{(by (9))}.
\end{aligned} \tag{13}$$

In other words,

$$K^\sigma(q) = -\frac{1}{2\lambda^\sigma(0,0)}\triangle\log\lambda^\sigma\Big|_{(0,0)} \leq -1$$

which proves (i).

If $K^\sigma \equiv -1$, then we must have an equalities in (13). In particular, we have an equality in Jensen's inequality (9) which implies that $\log f$ must be a constant, say c. Thus, $\lambda = e^c\omega$. Furthermore,

$$\int_{\tilde S}\log\lambda(x,y)\ \triangle\eta(r)dxdy = 2\int_{\tilde S}\lambda(x,y)\eta(x,y)dxdy$$

which then implies $e^c = \kappa$ and hence $G = h$. This proves (ii).

We are left to prove (iii). For a non-negative $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
\int_{\tilde S}(\triangle\varphi)\log f\ dxdy &= \int_{\tilde S}(\triangle\varphi)\log\lambda\ dxdy - \int_{\tilde S}(\triangle\varphi)\log\omega\ dxdy \\
&\geq 2\int_{\tilde S}\varphi\,(\lambda - \kappa\omega)\ dxdy,
\end{aligned}$$

where the last inequality follows from Theorem 11 and equation (5) after integrating by parts.

For $\tau \in (0, \sigma]$, choose a test function φ which approximates the characteristic function of $\mathbb{B}_\tau(0)$. We thus obtain

$$\int_{\partial \mathbb{B}_\tau(0)} \frac{\partial}{\partial r} \log f d\Sigma \geq 2 \int_{\mathbb{B}_\tau(0)} \lambda - \kappa\omega \;\; dxdy.$$

Using polar coordinates (r, θ),

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\int_0^{2\pi} \log f(r,\theta)\ d\theta\right)\Bigg|_{r=\tau} &= \int_0^{2\pi} \frac{\partial}{\partial r} \log f(r,\theta)\Bigg|_{r=\tau} d\theta \\
&= \frac{1}{\tau}\int_0^{2\pi} \frac{\partial}{\partial r} \log f(r,\theta)\Bigg|_{r=\tau} \tau d\theta \\
&\geq \frac{2}{\tau}\int_0^{2\pi}\int_0^{\tau} \lambda - \kappa\omega \ r dr d\theta.
\end{aligned}$$

Thus, for $\rho, t \in (0, \sigma)$ and $t < \rho$,

$$\begin{aligned}
&\int_0^{2\pi} \log f(\rho,\theta)\ d\theta - \int_0^{2\pi} \log f(t,\theta)\ d\theta \\
&= \int_t^{\rho} \frac{\partial}{\partial r}\left(\int_0^{2\pi} \log f(r,\theta)\ d\theta\right)\Bigg|_{r=\tau} d\tau \\
&\geq \int_t^{\rho} \frac{2}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\ dxdyd\tau.
\end{aligned}$$

Multiply the above inequality by t and integrate with respect to t over interval $[0, s]$ where $s < \rho$ to obtain

$$\begin{aligned}
&\frac{s^2}{2}\int_0^{2\pi} \log f(\rho,\theta)\ d\theta - \int_{\mathbb{B}_s(0)} \log f(t,\theta)\ dxdy \\
&= \int_0^{s} t \int_0^{2\pi} \log f(r,\theta)\ d\theta dt - \int_0^{s}\int_0^{2\pi} \log f(t,\theta)\ t d\theta dt \\
&\geq \int_0^{s} t \int_t^{\rho} \frac{2}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\ dxdyd\tau dt.
\end{aligned}$$

Since we only need to prove the inequality of (iii) for a.e. $q \in \tilde{S}$, we can assume without the loss of generality that q is a Lebesgue point for the integrable function $\log \lambda$. Thus dividing the above inequality by πs^2 and letting $s \to 0$, we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log f(r,\theta)\, d\theta - \log f(q) \\
&\quad \geq \lim_{s\to 0} \frac{1}{\pi s^2}\int_0^s t \int_t^\rho \frac{2}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\, dxdyd\tau dt \\
&\quad = \lim_{s\to 0} \frac{1}{\pi s^2}\int_0^s t \int_0^\rho \frac{2}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\, dxdyd\tau dt \\
&\qquad - \lim_{s\to 0} \frac{1}{\pi s^2}\int_0^s t \int_0^t \frac{2}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\, dxdyd\tau dt \\
&\quad = \frac{1}{2\pi}\int_0^\rho \frac{2}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\, dxdyd\tau \\
&\qquad - \lim_{s\to 0} \frac{1}{\pi s^2}\int_0^s t \int_0^t \frac{2}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\, dxdyd\tau dt. \qquad (14)
\end{aligned}$$

Since $|\lambda - \kappa\omega|$ is a bounded function in $\mathbb{B}_\sigma(0)$, we conclude that given $\epsilon > 0$, there exists $t_0 > 0$ sufficiently small such that

$$t \in (0, t_0) \;\Rightarrow\; \int_0^t \frac{2}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\, dxdyd\tau < \epsilon.$$

Thus

$$s \in (0, t_0) \;\Rightarrow \frac{1}{\pi s^2}\int_0^s t \int_0^t \frac{2}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\, dxdyd\tau dt \leq \frac{\epsilon}{2\pi}.$$

We can now conclude that the limit that appears in (14) is equal to 0, and therefore

$$\int_0^{2\pi} \log f(r,\theta)\, d\theta \geq 2\pi \log f(q) + \int_0^\rho \frac{2}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\, dxdyd\tau$$

By the definition of f^σ, we have

$$f^\sigma(q) = \exp \int_0^\sigma \left(\int_0^{2\pi} \log f(r,\theta) d\theta\right) \eta(r)\omega(r) r dr.$$

Combining the above,

$$f^\sigma(q) \geq f(q) \exp \int_0^\sigma \left(\int_0^\rho \frac{1}{\tau}\int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega)\, dxdyd\tau\right) \eta(r)\omega(r)\, rdr. \qquad (15)$$

The function

$$\psi : \tilde{S} \to [0, \infty), \quad \psi(q) := \int_0^\sigma \left(\int_0^\rho \frac{1}{\tau} \int_{\mathbb{B}_\tau(0)} \lambda \; dxdyd\tau \right) \eta(r)\omega(r) \; rdr$$

is continuous by Theorem 11 (i). Furthermore, $\psi(q) > 0$ for all $q \in \tilde{S}$ since Theorem 11 (iv) implies

$$\int_{\mathbb{B}_\tau(0)} \lambda \; dxdy > 0, \quad \forall \tau > 0.$$

Since ψ is also equivariant, this implies a positive lower bound for ψ; i.e. $\min_{q \in \tilde{S}} \psi(q) > 0$. Moreover, the function

$$q \mapsto \int_0^\sigma \left(\int_0^\rho \frac{1}{\tau} \int_{\mathbb{B}_\tau(0)} \omega \; dxdyd\tau \right) \eta(r)\omega(r) \; rdr$$

is constant. Thus, we can choose $\kappa > 0$ sufficiently small such that

$$\int_0^\sigma \left(\int_0^\rho \int_{\mathbb{B}_\tau(0)} (\lambda - \kappa\omega) \; dxdyd\tau \right) \eta(r)\omega(r) \; rdr \geq 0.$$

Thus, (15) implies (iii). Q.E.D.

4 Relation with Entropy

The purpose of this section is to state an inequality connecting the Gaussian curvature of the pullback metric of a ρ-equivariant conformal harmonic map with the with the action of ρ at infinity. This relationship was explored earlier by Sanders and the argument below is based on [San].

Definition 14 Let Γ be a subgroup of $\mathrm{Isom}(X)$. We define

$$\delta(\Gamma) := \lim_{R \to \infty} \frac{\log N_\Gamma(R, P)}{R}$$

where

$$N_\Gamma(R, P) = \left| \{ \gamma \in \Gamma : d(P, \gamma P) < R \} \right|.$$

The number $\delta(\Gamma)$ is independent of $P \in X$ and is a measure of the dynamical complexity of the group Γ.

Definition 15 Let Γ be a subgroup of Isom(X). The *limit set* Λ_Γ of Γ is the set of accumulation point of Γ-orbits of a fixed point $P \in X$ in the visual boundary $\partial_\infty X$. Equivalently, Λ_Γ is the smallest non-empty, closed Γ-invariant subset of $\partial_\infty X$.

Theorem 16 ([Co]) *If $\Gamma \subset Isom(X)$ is a discrete, convex cocompact subgroup, then*

$$\delta(\Gamma) = \dim_{\mathcal{H}}(\Lambda_\Gamma).$$

The Hausdorff dimension is computed using any of the Gromov metrics on $\partial_\infty X$.

Definition 17 The *volume entropy* of a smooth Riemannian metric G^σ on $\tilde{S}$ is

$$e(G^\sigma) = \lim_{R\to\infty} \frac{\log \mathrm{Vol}_{G^\sigma}(B_R^{G^\sigma}(x))}{R}.$$

where $B_R^{G_\sigma}(x)$ is the geodesic ball of radius R centered at x. The existence of the limit in Definition 17 is proven in [Man1] Section 2.

Remark 18 For $s > 0$ and $x, y \in \tilde{S}$, the critical exponent for the Poincaré series

$$P^s(x,y) = \sum_{\gamma\in\pi_1(\tilde{S})} e^{-sd_{G^\sigma}(x,\gamma y)}$$

is (cf. [Co] Proposition 5.3)

$$\delta(\pi_1(\tilde{S})) = \lim_{R\to\infty} \frac{\log N^{G^\sigma}_{\pi_1(S)}(R,x)}{R} = \lim_{R\to\infty} \frac{\log \#\{\gamma \in \Gamma : d_{G^\sigma}(\gamma x, x) \leq R\}}{R}.$$

On the other hand, for $(\tilde{S}, G_\sigma)$ as in Lemma 13, [Kn] Theorem 5.1 and Corollary 5.2 shows that the Poincaré series $P^s(x,y)$ converges for $s > e(G^\sigma)$ and diverges for $s \leq e(G^\sigma)$. In particular, we observe that

$$e(G^\sigma) = \lim_{R\to\infty} \frac{\log N^{G^\sigma}_{\pi_1(S)}(R,x)}{R}. \tag{16}$$

Theorem 19 ([Man1]) *For a smooth Riemannian metric G^σ on $\tilde{S}$,*

$$h_{top}\left(\frac{G^\sigma}{Vol_{G^\sigma}(S)}\right) \leq e\left(\frac{G^\sigma}{Vol_{G^\sigma}(S)}\right)$$

where h_{top} is the topological entropy of the geodesic flow with respect to the Liouville measure. Moreover,

$$K_{G^\sigma} \leq 0 \;\Rightarrow\; h_{top}\left(\frac{G^\sigma}{Vol_{G^\sigma}(S)}\right) = e\left(\frac{G^\sigma}{Vol_{G^\sigma}(S)}\right)$$

where K_{G^σ} is the Gauss curvature of the smooth metric G^σ.

Theorem 20 ([Man2]) *For a smooth Riemannian metric G^σ on $\tilde{S}$,*

$$\frac{1}{\sqrt{Vol_{G^\sigma}(S)}}\int_S \sqrt{-K_{G^\sigma}}d\mu_{G^\sigma} \leq h^L_{G^\sigma}$$

where $h^L_{G^\sigma}$ is the metric entropy of the geodesic flow with respect to Liouville measure.

Lemma 21 *If G is the pullback metric on $\tilde{S}$ associated to a ρ-equivariant conformal harmonic map $\tilde{u} : \mathbb{H}^2 \to X$ and G^σ, K_{G^σ} are as in Lemma 13, then*

$$\frac{1}{Vol_{G^\sigma}(S)}\int_S \sqrt{-K_{G^\sigma}}d\mu_{G^\sigma} \leq \dim_{\mathcal{H}}(\Lambda_\Gamma).$$

Proof. Since G^σ is a smooth metric of negative Gaussian curvature on $\tilde{S}$,

$$\begin{aligned}
\frac{1}{\sqrt{Vol_{G^\sigma}(S)}}\int_S \sqrt{-K_{G^\sigma}}d\mu_{G^\sigma} &\leq h^L_{G^\sigma} \quad \text{(by Theorem 20, [Man2])} \\
&\leq h_{top}\left(\frac{G^\sigma}{Vol_{G^\sigma}(S)}\right) \quad \text{(by the variational principle)} \\
&= e\left(\frac{G^\sigma}{Vol_{G^\sigma}(S)}\right) \quad \text{(by Theorem 19, [Man1])} \\
&= \sqrt{Vol_{G^\sigma}(S)}e(G^\sigma).
\end{aligned}$$

Thus,

$$\frac{1}{Vol_{G^\sigma}(S)}\int_S \sqrt{-K_{G^\sigma}}d\mu_{G^\sigma} \leq e(G^\sigma). \tag{17}$$

Fix $x \in \tilde{S}$. Recall that

$$N^{G^\sigma}_{\pi_1(S)}(R, x) := \{\gamma \in \pi_1(S) : d_{G^\sigma}(x, \gamma x) \leq R\}.$$

Using the fact that G is the pullback metric of $\tilde{u}$ together with Lemma 13 assertion (iii), we conclude

$$d(\tilde{u}(x), \rho(\gamma)\tilde{u}(x)) \leq d_G(x, \gamma x) \leq d_{G^\sigma}(x, \gamma x), \quad \forall x \in \tilde{S}.$$

Thus, letting $\Gamma = \rho(\pi_1(S))$, we have

$$N^{G^\sigma}_{\pi_1(S)}(R, x) \leq N_{\rho(\pi_1(S))}(R, \tilde{u}(x)).$$

Take logarithm, divide by R and let $R \to \infty$ in the above inequality. We then obtain $e(G^\sigma)$ on the left hand side by (16). On the right hand side, apply Theorem 16 to obtain

$$\lim_{R\to\infty} \frac{N_{\rho(\pi_1(S))}(R, \tilde{u}(x))}{R} = \delta(\Gamma) = \dim_{\mathcal{H}}(\Lambda_\Gamma).$$

We thus obtain

$$e(G^\sigma) \leq \dim_{\mathcal{H}}(\Lambda_\Gamma). \tag{18}$$

Combining (17) and (18) yields

$$\frac{1}{Vol_{G^\sigma}(S)} \int_S \sqrt{-K_{G^\sigma}} d\mu_{G^\sigma} \leq \dim_{\mathcal{H}}(\Lambda_\Gamma).$$

Q.E.D.

5 Proof of Theorem 1

Proof of Theorem 1. Let λ be the conformal factor of $\tilde{u}$, $G = \lambda(dx^2 + dy^2)$ as in Definition 12 and G^σ as in Lemma 21. By Lemma 21 and the assumption that $\dim_{\mathcal{H}}(\Lambda_\Gamma) = 1$, we have

$$\int_S \sqrt{-K_{G^\sigma}} d\mu_{G^\sigma} \leq Vol_{G^\sigma}(S)$$

Since $K_{G^\sigma} \leq -1$, we thus conclude $K_{G^\sigma} \equiv -1$. By Lemma 13 (ii), G is the hyperbolic metric h.

For any $p \in \tilde{S}$, identify $p = 0$ via normal coordinates and let (r, θ) be polar coordinates. For $R > 0$, define $\phi : B_R(p) \to X$ by setting

$$\phi(r, \theta) = (1 - r)\, \tilde{u}(p) + r\tilde{u}(1, \theta)$$

where, using a common notation in NPC geometry, the sum on the right hand side denotes the geodesic interpolation. In other words, ϕ maps geodesics emanating from p to geodesics emanating from $\tilde{u}(p)$. Therefore,

$$\left|\frac{\partial \phi}{\partial r}\right|^2 = d^2(\tilde{u}(1,\theta), \tilde{u}(p)) \leq d_G^2((1,\theta), p). \tag{19}$$

We next claim that

$$\left|\frac{\partial \tilde{u}}{\partial r}\right|^2 (r,\theta) = \lim_{\epsilon \to 0} \frac{d_G^2((r,\theta),(r+\epsilon,\theta))}{\epsilon^2} \tag{20}$$

Indeed, since G is equal to the hyperbolic metric h and (r,θ) are polar coordinates for h

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{d_G^2((r,\theta),(r+\epsilon,\theta))}{\epsilon^2} &= \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} \int_r^{r+\epsilon} \sqrt{\lambda}(s,\theta) ds \right)^2 \\ &= \lambda(r,\theta) = \left|\frac{\partial \tilde{u}}{\partial r}\right|^2 (r,\theta) \end{aligned}$$

the last equality being because $\tilde{u}$ is conformal. On the other hand, again since (r,θ) are polar coordinates for $h = G$

$$\lim_{\epsilon \to 0} \frac{d_G^2((r,\theta),(r+\epsilon,\theta))}{\epsilon^2} = d_G^2((1,\theta), p). \tag{21}$$

Combining equalities (19), (20) and (21) we obtain

$$\left|\frac{\partial \phi}{\partial r}\right|^2 (r,\theta) \leq \left|\frac{\partial \tilde{u}}{\partial r}\right|^2 (r,\theta). \tag{22}$$

Furthermore,

$$d(\phi(1,\theta_1), \phi(1,\theta_2)) = d(u(1,\theta_1), u(1,\theta_2)) \leq d_G((1,\theta_1),(1,\theta_2)).$$

Since we have shown that G is the hyperbolic metic, the CAT(-1) condition implies that

$$d(\phi(r,\theta_1), \phi(r,\theta_2)) \leq d_G((r,\theta_1),(r,\theta_2)).$$

Thus,

$$\begin{aligned} \left|\frac{\partial \phi}{\partial \theta}\right|^2 (r,\theta) &= \lim_{\epsilon \to 0} \frac{d^2(\phi(r,\theta), \phi(r,\theta+\epsilon))}{\epsilon^2} \\ &\leq \lim_{\epsilon \to 0} \frac{d_G^2((r,\theta),(r,\theta+\epsilon))}{\epsilon^2} = \left|\frac{\partial \tilde{u}}{\partial \theta}\right|^2 (r,\theta). \end{aligned} \tag{23}$$

Notice that the derivation of the last equality is similar to that of (20). Thus (22) and (23) imply that $E^{\phi} \leq E^{\tilde{u}}$, but since $\tilde{u}$ is energy minimizing $\phi = \tilde{u}$. Therefore, $\tilde{u}$ maps radial lines emanating from p to geodesics. Since p is an arbitrary point in $\tilde{S}$, this proves $\tilde{u}$ is totally geodesic. Q.E.D.

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