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Essential regularity of the model space for the Weil–Petersson metric

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Abstract. This is the second in a series of papers ([7] and [6] are the others) that studies the behavior of harmonic maps into the Weil–Petersson completion $\overline{\mathcal{T}}$ of Teichmüller space. The boundary of $\overline{\mathcal{T}}$ is stratified by lower-dimensional Teichmüller spaces and the normal space to each stratum is a product of copies of a singular space $\overline{\mathbf{H}}$ called the model space. The significance of $\overline{\mathbf{H}}$ is that it captures the singular behavior of the Weil–Petersson geometry of $\overline{\mathcal{T}}$. The main result of the paper is that certain subsets of $\overline{\mathbf{H}}$ are *essentially regular* in the sense that harmonic maps to those spaces admit uniform approximation by affine functions. This is a modified version of the notion of essential regularity introduced by Gromov–Schoen in [12] for maps into Euclidean buildings and is one of the key ingredients in proving superrigidity. In the process, we introduce new coordinates on $\overline{\mathbf{H}}$ and estimate the metric and its derivatives with respect to the new coordinates. These results form the technical core for studying the analytic behavior of harmonic maps into the completion of Teichmüller space and are utilized in our subsequent paper [6], where we prove the holomorphic rigidity of the Teichmüller space and several rigidity results for the mapping class group.

1. Introduction

Let $\mathcal{T}$ denote the Teichmüller space of a genus g Riemann surfaces with n punctures and $3g - 3 + n > 0$. Recall that the cotangent space $T^*_{[\sigma]}\mathcal{T}$ of $\mathcal{T}$ at a conformal structure defined by the hyperbolic metric σ can be identified with the space of holomorphic quadratic differentials with respect to $[\sigma]$. The Weil–Petersson co-metric of the quadratic differential $\phi \in T^*_{[\sigma]}\mathcal{T}$ is defined as L^2-norm of ϕ with respect to σ. This induces an incomplete, smooth Riemannian metric in $\mathcal{T}$ of non-positive sectional curvature called the *Weil–Petersson metric* (cf. [1, 2, 18, 19, 21] among numerous references). The metric completion $\overline{\mathcal{T}}$ of $\mathcal{T}$ is no longer a Riemannian manifold, but instead it is an *NPC space* or otherwise known as a CAT(0) space; i.e. a complete metric space of non-positive curvature in the sense of Alexandrov (cf. [23]).

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In our previous work (cf. [5, 7, 9]), we studied rigidity problems in group theory via harmonic maps into certain NPC spaces. In this paper and its sequel [6], we study rigidity properties of Teichmüller space and the mapping class group via harmonic maps into the NPC space $\overline{\mathcal{T}}$. The goal is to prove the *holomorphic rigidity conjecture for Teichmüller space* (first stated in [13]) which, loosely stated, says that *the mapping class group uniquely determines the Teichmüller space as a complex manifold*. The key step is to show that the singular set of a harmonic map into $\overline{\mathcal{T}}$ is of Hausdorff codimension at least 2. Since $\mathcal{T}$ is a smooth manifold, the singular set of a harmonic map into $\overline{\mathcal{T}}$ is the set of points that is mapped into $\partial\mathcal{T} = \overline{\mathcal{T}} \backslash \mathcal{T}$.

The Weil–Petersson geometry near the boundary $\partial\mathcal{T}$ has been studied by various authors (cf. [22] and references therein). The boundary $\partial\mathcal{T}$ is stratified by lower-dimensional Teichmüller spaces with each stratum being geodesically convex. Furthermore, in a neighborhood $\mathcal{N}$ of a boundary point, the Weil–Petersson metric can be approximated up to higher order by a simpler model space. Indeed, $\mathcal{N}$ is asymptotically a product $\mathcal{U} \times \mathcal{V}$ (cf. [8, 10, 22, 23]), where $\mathcal{U}$ is an open subset of a lower-dimensional Teichmüller space along with the Weil–Petersson metric and $\mathcal{V}$ is an open subset of $\overline{\mathbf{H}} \times \cdots \times \overline{\mathbf{H}}$, where $\overline{\mathbf{H}}$ is the *model space*.

To define the model space $\overline{\mathbf{H}}$, let

$$\mathbf{H} = \{(\rho, \phi) \in \mathbb{R}^2 : \rho > 0\} \quad \text{and} \quad g_{\mathbf{H}}(\rho, \phi) = d\rho^2 + \rho^6 d\phi^2. \tag{1}$$

(Note that in most literature on Weil–Petersson geometry, one considers the slightly different metric $4dr^2 + r^6 d\theta^2$ which is clearly isometric to $g_{\mathbf{H}}$ via the change of coordinates $\rho = 2r, \phi = \frac{\theta}{8}$.) The Christoffel symbols of this metric are

$$\begin{aligned} \Gamma^{\rho}_{\rho\rho} &= 0, & \Gamma^{\phi}_{\phi\phi} &= 0, \\ \Gamma^{\rho}_{\rho\phi} &= 0 & \Gamma^{\phi}_{\rho\phi} &= \frac{3}{\rho}, \\ \Gamma^{\rho}_{\phi\phi} &= -3\rho^5, & \Gamma^{\phi}_{\rho\rho} &= 0 \end{aligned} \tag{2}$$

and the Gauss curvature is given by

$$K = -\frac{6}{\rho^2}.$$

The geodesic equations for $\gamma = (\gamma_\rho, \gamma_\phi)$ in terms of the coordinates (ρ, ϕ) are given by

$$\gamma_\rho \triangle \gamma_\rho = 3\gamma_\rho^6 |\nabla\gamma_\phi|^2 \quad \text{and} \quad \gamma_\rho^4 \triangle \gamma_\phi = -6\nabla\gamma_\rho \cdot \gamma_\rho^3 \nabla\gamma_\phi. \tag{3}$$

The Riemannian manifold $(\mathbf{H}, g_{\mathbf{H}})$ is not complete; indeed, for a fixed $\phi_0 \in \mathbb{R}$, the geodesic $\gamma = (\gamma_\rho, \gamma_\phi) : [0, 1) \to \mathbf{H}$ given by

$$\gamma_\rho(t) = 1 - t, \quad \gamma_\phi(t) = \phi_0$$

leaves every compact subset of $\mathbf{H}$ and has a length of 1. The incompleteness of $\mathcal{T}$ was exhibited by Wolpert [21] and Chu [4]; indeed certain curves that leave every compact set have finite length with respect to the Weil–Petersson metric. These curves correspond to deformations of compact Riemann surfaces via neck pinching in which nontrivial loops degenerate to nodes. The metric completion of $(\mathbf{H}, g_{\mathbf{H}})$ is constructed by identifying the axis $\rho = 0$ to a single point P_0 and setting

$$\overline{\mathbf{H}} = \mathbf{H} \cup \{P_0\}.$$

The distance function $d_{\mathbf{H}}$ induced by $g_{\mathbf{H}}$ is extended to $\overline{\mathbf{H}}$ by setting $d_{\mathbf{H}}(Q, P_0) = \rho$ for $Q = (\rho, \phi) \in \mathbf{H}$. The following facts are easy to check (cf. [10]):

(1) The Riemannian surface $(\mathbf{H}, g_{\mathbf{H}})$ is geodesically convex.

(2) The complete metric space $(\overline{\mathbf{H}}, d_{\mathbf{H}})$ is an NPC space.

(3) The space $(\overline{\mathbf{H}}, d_{\mathbf{H}})$ is not locally compact.

Since each boundary stratum of $\overline{\mathcal{T}}$ is a smooth Riemannian manifold, the singular behavior of the Weil–Petersson geometry is completely captured by the model space $\overline{\mathbf{H}}$. For one, the Gauss curvature of $\mathbf{H}$ approaching $-\infty$ near its boundary $\{P_0\}$ reflects the sectional curvature blow-up of $\mathcal{T}$ near $\partial\mathcal{T}$. Moreover, the non-local compactness of $\overline{\mathcal{T}}$ is also captured by $\overline{\mathbf{H}}$. Indeed, a geodesic ball in $\overline{\mathbf{H}}$ centered at P_0 is not compact; for example, the points in the sequence $\{(r_0, n) \subset \overline{\mathbf{H}} : n = 1, 2, \dots\}$ are at a fixed distance r_0 from P_0, and yet the sequence does not have a converging subsequence. Finally, $\overline{\mathbf{H}}$ does not have a geodesic extendability property; i.e. not every geodesic segment can be extended to a geodesic line. In particular, the geodesic $\gamma : [0, 1) \to \mathbf{H}$ given above cannot be extended to be defined on $[0, T]$ for $T > 1$. The inability to extend geodesics also characterizes the Weil–Petersson geometry of Teichmüller space. Many important results concerning the geometry of non-positively curved Riemannian manifold spaces rely in an essential way on having the geodesic extension property. (We refer to [3] for results in this direction for NPC spaces with geodesic extendability.)

In order to analyze the behavior of harmonic maps into $\overline{\mathcal{T}}$, we first need to analyze the behavior of harmonic maps into $\overline{\mathbf{H}}$, and the goal of this paper is to develop tools to study harmonic map theory into $\overline{\mathbf{H}}$. To this end, we start with an important observation that a harmonic map $u : \Omega \to \overline{\mathbf{H}}$ can be closely approximated near an order one singular point by a harmonic map into

$$\overline{\mathbf{H}}_2 = \overline{\mathbf{H}}^+ \sqcup \overline{\mathbf{H}}^- /\!\sim$$

where $\overline{\mathbf{H}}^+$ and $\overline{\mathbf{H}}^-$ denote two distinct copies of $\overline{\mathbf{H}}$ and $\sim$ indicates that the point P_0 from each copy is identified as a single point. To understand this in the case of $\overline{\mathbf{H}}_2$, note that because of the non-local compactness of $\overline{\mathbf{H}}$ near P_0, the Alexandrov tangent space $T_{P_0}\overline{\mathbf{H}}$ of $\overline{\mathbf{H}}$ at P_0 (which is isometric to the interval $[0, \infty)$) does not properly reflect the geometry of $\overline{\mathbf{H}}$ in a neighborhood of P_0. Indeed, if there exists a harmonic map $u : B_1(0) \to \overline{\mathbf{H}}$ with $u(0) = P_0$, then its tangent map at 0 does not map into $T_{P_0}\overline{\mathbf{H}}$. Instead, its image can be embedded into a metric space constructed by joining a multiple copies of $\overline{\mathbf{H}}$ at P_0 (cf. [20]). In particular, a tangent map at an order 1 point is isometric to $\mathbf{R}$ and can be embedded in $\overline{\mathbf{H}}_2$ as a geodesic that hits the singular point P_0 (cf. Lemma 10). These results are presented in Section 2 with many proofs deferred to Section 7 in the Appendix. An explanation of this phenomenon is outlined in Observation 1 below. We also note that the idea of supplying an extra copy of Teichmüller space across each Weil–Petersson boundary stratum appeared in [24] where using a result of [17], the resulting space is again NPC.

Motivated by the phenomenon discussed in the last paragraph, we first consider harmonic maps into $\overline{\mathbf{H}}_2$. To study these maps, we take advantage of the fact that $\overline{\mathbf{H}}_2$ is almost a complete Riemannian manifold; more precisely, $\overline{\mathbf{H}}_2$ has the geodesic extension property and $\overline{\mathbf{H}}_2 \backslash \{P_0\}$ is an union of two Riemannian manifolds $\mathbf{H}^+$ and $\mathbf{H}^-$. In this sense, we are in a setting similar to that of Gromov and Schoen's foundational paper [12] that initiated the theory of harmonic map into singular spaces. There, Gromov and Schoen study harmonic maps into Euclidean buildings. (More generally, they consider F-connected complexes. Also relevant are [7] and [9] where the authors study harmonic maps into hyperbolic buildings.) A Euclidean building can be viewed as a union of copies of equidimensional Euclidean spaces with each copy geodesically convex. This viewpoint plays an important role in the regularity theory of harmonic maps; indeed, each copy of a Euclidean space in a Euclidean building is an example of a subspace called *essentially regular* in [12]. We refer to Section 5 of that paper for the precise definition of essentially regular subspaces of an NPC space. Here, we shall summarize the notion of an essentially regular space by the characterization that any harmonic maps into it is well-approximated near a point by an affine map and this *approximation is uniform* in the sense that the approximation depends locally only on the total energy of the harmonic map. The existence of many essentially regular subsets is the key ingredient in proving regularity of harmonic maps (cf. [12, Theorem 5.1]), which implies the non-Archimedean rank 1 superrigidity theorem.

As the first step toward the regularity theory of harmonic maps into $\overline{\mathcal{T}}$, we will prove the existence of many essentially regular geodesically convex subsets of $\overline{\mathbf{H}}_2$ containing the point P_0. We note that we have so far been unable to prove that these subsets are essentially regular in the strict sense of Gromov–Schoen [12]. On the other hand, we will prove a weaker notion of essentially regular that is sufficient for obtaining good estimates for harmonic maps in the sequel [6]. For convenience, we will *also call this weaker notion essentially regular*. (We remark that the proof of essentially regularity of smooth Riemannian manifold in [12, Section 5] appears to be incorrect, and we have so far been unable to correct it. Thus, as far as we know, Euclidean spaces and buildings are the only known examples of essentially regular sets in the strict sense of [12]. We emphasize that this plays no role in the rest of [12] and *does not affect the validity of the other results of* [12].) Given that the local geometry of $\overline{\mathbf{H}}_2$ is very singular near P_0, it is somewhat surprising that essentially regular subsets containing P_0 exist in $\overline{\mathbf{H}}_2$. By a similar argument, we can also prove the existence of essentially regular subsets arbitrarily close to P_0 in $\overline{\mathbf{H}}$. The idea for this is to relate the geometry of $\overline{\mathbf{H}}_2$ and $\overline{\mathbf{H}}$ near the singular point P_0.

We now discuss the how we will relate the geometry of $\overline{\mathbf{H}}_2$ and $\overline{\mathbf{H}}$. First, note that the coordinates for $\overline{\mathbf{H}}_2\backslash\{P_0\}$ are given by $(\rho,\phi)\in\mathbb{R}^2\backslash\{\rho=0\}$ with $(\rho,\phi)\in\mathbf{H}^+$ when $\rho\in(0,\infty)$ and the point $(-\rho,\phi)\in\mathbf{H}^-$ when $\rho\in(-\infty,0)$. Let $g_{\mathbf{H}_2}$ be the Riemannian metric on $\overline{\mathbf{H}}_2\backslash\{P_0\}$ inherited from $(\mathbf{H}, g_{\mathbf{H}})$. With respect to the coordinates (ρ,ϕ) we have

$$g_{\mathbf{H}_2} = d\rho^2 + \rho^6 d\phi^2. \tag{4}$$

The induced distance function $d_{\mathbf{H}_2}$ on $\overline{\mathbf{H}}_2$ is given by

$$d_{\mathbf{H}_2}((\rho_1,\phi_1),(\rho_2,\phi_2)) = \begin{cases} d_{\mathbf{H}}((|\rho_1|,\phi_1),(|\rho_2|,\phi_2)) & \text{if } \rho_1\rho_2 \geq 0,\\ |\rho_1|+|\rho_2| & \text{if } \rho_1\rho_2 < 0.\end{cases}$$

The Christoffel symbols with respect to this metric are given again by (2) and the Gauss curvature $\lim_{\rho\to 0} K = -\infty$, making the geometry of $\overline{\mathbf{H}}_2$ very singular near P_0. The fundamental relation between the geometry of the spaces $\overline{\mathbf{H}}$ and $\overline{\mathbf{H}}_2$ near P_0 is captured in the following observation which plays a key role for the rest of the paper.

Observation 1. *Let* $\phi_0 > 0$ *and let* $\sigma^{\phi_0} = (\sigma^{\phi_0}_\rho, \sigma^{\phi_0}_\phi) : (-\infty,\infty) \to \overline{\mathbf{H}}$ *be a piecewise geodesic defined by* $\sigma^{\phi_0}(s) = (s,\phi_0)$ *for* $s\in[0,\infty)$ *and* $\sigma^{\phi_0}(s) = (-s,-\phi_0)$ *for* $s\in(-\infty,0]$. *Let* $\gamma^{\phi_0} = (\gamma^{\phi_0}_\rho,\gamma^{\phi_0}_\phi) : (-\infty,\infty)\to\mathbf{H}$ *be the unit speed geodesic passing through the points*

$$Q^{\phi_0}_- = (1,-\phi_0) \quad \textit{and} \quad Q^{\phi_0}_+ = (1,\phi_0).$$

Then

$$d(\gamma^{\phi_0}, \sigma^{\phi_0}) \to 0 \quad \text{as } \phi_0 \to \infty \text{ uniformly on the interval } [-1, 1]. \tag{5}$$

To prove of this claim, we first make three subclaims:

(i) *For any* ϕ_0, $\gamma_\rho^{\phi_0}(0) \le \gamma_\rho^{\phi_0}(s)$ *for all* $s \in (-\infty, \infty)$.

(ii) $\gamma_\rho^{\phi_0}(0) \to 0$ *as* $\phi_0 \to \infty$.

(iii) $d(Q_-^{\phi_0}, \gamma^{\phi_0}(-1)) = d(Q_+^{\phi_0}, \gamma^{\phi_0}(1)) \to 0$ *as* $\phi_0 \to 0$.

Proof. (i) The geodesic equations

$$\gamma_\rho^{\phi_0} \frac{d^2\gamma_\rho^{\phi_0}}{ds^2} = 3(\gamma_\rho^{\phi_0})^6 \left(\frac{d\gamma_\phi^{\phi_0}}{ds}\right)^2 \quad \text{and} \quad (\gamma_\rho^{\phi_0})^4 \frac{d^2\gamma_\phi^{\phi_0}}{ds^2} = -6(\gamma_\rho^{\phi_0})^3 \frac{d\gamma_\rho^{\phi_0}}{ds}\frac{d\gamma_\phi^{\phi_0}}{ds}$$

imply $\gamma_\rho^{\phi_0}$ is convex. Combining this with the symmetry of $\gamma_\rho^{\phi_0}$, subclaim (i) follows.

(ii) If $\gamma_\rho^{\phi_0}(0) \ge c > 0$, then $\gamma_\rho^{\phi_0}(s) \ge c$ by subclaim (i) and hence

$$1 = \left|\frac{d\gamma^{\phi_0}}{ds}\right|^2 = \left(\frac{d\gamma_\rho^{\phi_0}}{ds}\right)^2 + \gamma_\rho^6(s)\left(\frac{d\gamma_\phi^{\phi_0}}{ds}\right)^2 \ge c^6\left(\frac{d\gamma_\phi^{\phi_0}}{ds}\right)^2.$$

Thus,

$$\phi_0^2 = |\gamma_\phi^{\phi_0}(1)|^2 \le \left(\int_0^1 \left|\frac{d\gamma_\phi^{\phi_0}}{ds}\right| ds\right)^2 \le \int_0^1 \left|\frac{d\gamma_\phi^{\phi_0}}{ds}\right|^2 ds \le c^{-6}.$$

Since this impossible for large ϕ_0, we have proven subclaim (ii).

(iii) This assertion follows immediately from the fact that γ^ϕ is a unit speed geodesic passing through points $Q_-^{\phi_0}$ and $Q_+^{\phi_0}$ and that $d(Q_-^{\phi_0}, Q_+^{\phi_0}) \to 2$ as $\phi_0 \to \infty$.

Subclaims (ii) and (iii) assert

$$d(\sigma^{\phi_0}(0), \gamma^{\phi_0}(0)) = d(P_0, \gamma^{\phi_0}(0)) = \gamma_\rho^{\phi_0}(0) \to 0$$

and

$$d(\sigma^{\phi_0}(1), \gamma^{\phi_0}(1)) = d(Q_+^{\phi_0}, \gamma^{\phi_0}(1)) \to 0.$$

Since γ^{ϕ_0} and σ^{ϕ_0} are geodesics on the interval [0, 1], Claim (5) follows from the convexity of geodesics in an NPC space. □

The point of Observation 1 is to show that the geodesic of $\overline{\mathbf{H}}$ through $Q_\pm^{\phi_0}$ (such a geodesic is called a *symmetric geodesic*) is almost like two geodesics $\rho \mapsto (\rho, \pm\phi_0)$ for ϕ_0 large (a geodesic with constant ϕ-coordinate is called a *vertical geodesic*). One can get a hint of the usefulness of this behavior by noting that, given two vertical geodesics in $\overline{\mathbf{H}}$ starting at P_0, if we identify one of these geodesics in $\overline{\mathbf{H}}_+ \subset \overline{\mathbf{H}}_2$ and the other in $\overline{\mathbf{H}}_- \subset \overline{\mathbf{H}}_2$, we then obtain a geodesic in $\overline{\mathbf{H}}_2$. We will take advantage of Observation 1 by foliating $\overline{\mathbf{H}}$ by symmetric geodesics and comparing this to the foliation of $\overline{\mathbf{H}}_2$ by vertical geodesics. We will explain this in more detail later in this introduction.

The singular set of a harmonic map u from a Riemannian domain Ω to $\mathbf{H}$ or $\mathbf{H}_2$ is defined by

$$\mathcal{S}(u) = \{x \in \Omega : u(x) = P_0\}.$$

A singular point is a point in $S(u)$ and a regular point is a point that is not a singular point. Since $\overline{\mathbf{H}}_2 \backslash \{P\}$ is a smooth Riemannian manifold, for a harmonic map $u : \Omega \to \overline{\mathbf{H}}_2$, we can write in small neighborhood of a regular point $u = (u_\rho, u_\phi)$ in terms of the coordinates (ρ, ϕ). The harmonic map equations in these coordinates are

$$u_\rho \triangle u_\rho = 3u_\rho^6 |\nabla u_\phi|^2 \quad \text{and} \quad u_\rho^4 \triangle u_\phi = -6\nabla u_\rho \cdot u_\rho^3 \nabla u_\phi. \tag{6}$$

Although the right-hand side of the above equations is locally bounded by the Lipschitz regularity of harmonic maps (cf. [15, Theorem 2.4.6]), the left-hand side of the equations is degenerate since $u_\rho(x)$ is the distance of the image $u(x)$ to P_0 which tends to zero. Thus, from this point of view, it is hard to see why the map should be uniformly regular near a singular point. We remark that in order to fit our equations (6) into the existing framework of degenerate elliptic PDEs one needs to have a precise decay estimate for u_ρ which does not a priori exist. As first pointed out to us by P. Daskalopoulos and also exploited in the two-dimensional case in [20], one might try to approach this PDE by using results of Koch (cf. [14]), but so far we have not been able to apply these techniques successfully to our setting. On the other hand, we are aided here by the fact that our problem arises from a *geometric* problem; indeed, we are considering *minimizing* maps into the space $\overline{\mathbf{H}}_2$ of non-positive curvature. As illustrated in the next example, it turns out that some of the degeneracy of our PDEs is due to the fact that the natural choice of coordinates (ρ, ϕ) coming from Teichmüller theory is a bad choice of coordinates from the point of view of PDE:

Example 2. Consider the two-dimensional Euclidean space. One can use polar coordinates to express this space as a Riemannian manifold; namely, the Euclidean space is the Riemannian manifold $(\mathbf{H}, g_0)$, where

$$\mathbf{H} = \{(\rho, \phi) \in \mathbb{R}^2 : \rho > 0\} \quad \text{and} \quad g_0(\rho, \phi) = d\rho^2 + \rho^2 d\phi^2.$$

The Christoffel symbols with respect to the polar coordinates (ρ, ϕ) are

$$\begin{aligned} \Gamma^\rho_{\rho\rho} &= 0, & \Gamma^\phi_{\phi\phi} &= 0, \\ \Gamma^\rho_{\rho\phi} &= 0, & \Gamma^\phi_{\rho\phi} &= \frac{1}{\rho}, \\ \Gamma^\rho_{\phi\phi} &= -\rho, & \Gamma^\phi_{\rho\rho} &= 0. \end{aligned}$$

For a map u into the two-dimensional Euclidean space, write $u = (u_\rho, u_\phi)$ with respect to the polar coordinates (ρ, ϕ). Then the harmonic map equations are

$$u_\rho \triangle u_\rho = u_\rho^2 |\nabla u_\phi|^2 \quad \text{and} \quad u_\rho^2 \triangle u_\phi = -2\nabla u_\rho \cdot u_\rho \nabla u_\phi. \tag{7}$$

This equation looks very similar to equation (6) for harmonic maps into $(\mathbf{H_2}, g_{\mathbf{H}_2})$ (or $(\mathbf{H}, g_{\mathbf{H}})$). On the other hand, we can also write down the two-dimensional Euclidean space by using the usual Cartesian coordinates. Correspondingly, we have $\mathbb{R}^2$ and $g_0(x, y) = dx^2 + dy^2$. Writing $u = (u_x, u_y)$ with respect to the Cartesian coordinates (x, y), the harmonic maps equations in coordinates (x, y) are

$$\triangle u_x = 0 \quad \text{and} \quad \triangle u_y = 0. \tag{8}$$

One can go from (7) to (8) by the change of variables

$$(\rho, \phi) \mapsto (x = \rho \cos\phi, y = \rho \sin\phi). \tag{9}$$

Relevant to the techniques of this paper is that the smoothness of u_x and u_y can be immediately deduced from the theory of elliptic partial differential equations.

In Section 3, we prove the existence of subsets of $\overline{\mathbf{H}}_2$ that are essentially regular. The key is the introduction of new coordinates in $\overline{\mathbf{H}}_2$ that are motivated by the above example. Specifically, we let

$$\Upsilon := \rho - \frac{3}{2}\rho^5\phi^2 \quad \text{and} \quad \Phi := \rho^3\phi. \tag{10}$$

To explain the relevance of the new coordinates (Υ, Φ), we first consider $(\rho, \rho^2\phi)$ as analogues of polar coordinates (r, θ) of $\mathbb{R}^2$. Then the coordinates

$$(\rho, \phi) \mapsto (\rho \cos \sqrt{3}\rho^2\phi, \rho \sin \sqrt{3}\rho^2\phi)$$

are the analogues of the standard Euclidean coordinates (9). The coordinates Υ and $\sqrt{3}\Phi$ agree up to the first order with $\rho \cos \sqrt{3}\rho^2\phi$ and $\rho \sin \sqrt{3}\rho^2\phi$, respectively. We write $u = (u_\Upsilon, u_\Phi)$ in terms of coordinates (Υ, Φ) and study the harmonic map equations (cf. (32) and (33) below) to obtain regularity results. An important observation about Example 2 is the implicit use of the assumption $0 \le v_\theta < 2\pi$. (We need this assumption in order to show that the change of variables defines a diffeomorphism away from the origin.) In fact, without assuming this bound, it is unclear whether the solutions to (7) are regular. For a harmonic $u : \Omega \to \overline{\mathbf{H}}_2$, we do not have an a priori bound on the “angular” component function. But as mentioned before, the strategy for proving regularity of u is to first find almost essential regular subsets of $\overline{\mathbf{H}}_2$. This leads us to fix $\phi_0 > 0$ and to define

$$\overline{\mathbf{H}}_2[\phi_0] = \{(\rho, \phi) \in \overline{\mathbf{H}}_2 : |\phi| \le \phi_0\}.$$

Since $s \mapsto (s, \phi_0)$ and $s \mapsto (s, -\phi_0)$ are geodesics, $\overline{\mathbf{H}}_2[\phi_0]$ is geodesic convex in $\overline{\mathbf{H}}_2$. A harmonic map v whose image lies in $\overline{\mathbf{H}}_2[\phi_0]$ has the property that its “angular” component function v_ϕ is bounded. In Section 3, we use this to show that $\overline{\mathbf{H}}_2[\phi_0]$ is essentially regular. The precise statement is given in Theorem 15.

In Section 4, we explain the relationship between the geometries of $\overline{\mathbf{H}}$ and $\overline{\mathbf{H}}_2$ near P_0. First, observe that $\overline{\mathbf{H}}_2$ is foliated by an one-parameter family of geodesic lines $\{\rho \mapsto (\rho, \phi)\}$ (with parameter ϕ and a singularity at P_0). Motivated by this, we also foliate $\mathbf{H}$ by a families of geodesics. More specifically, we consider the one parameter family of geodesics

$$c = (c_\rho, c_\phi) : (-\infty, \infty) \times \left(-\infty, \frac{3}{2}\right) \to \mathbf{H} \tag{11}$$

satisfying the following conditions:

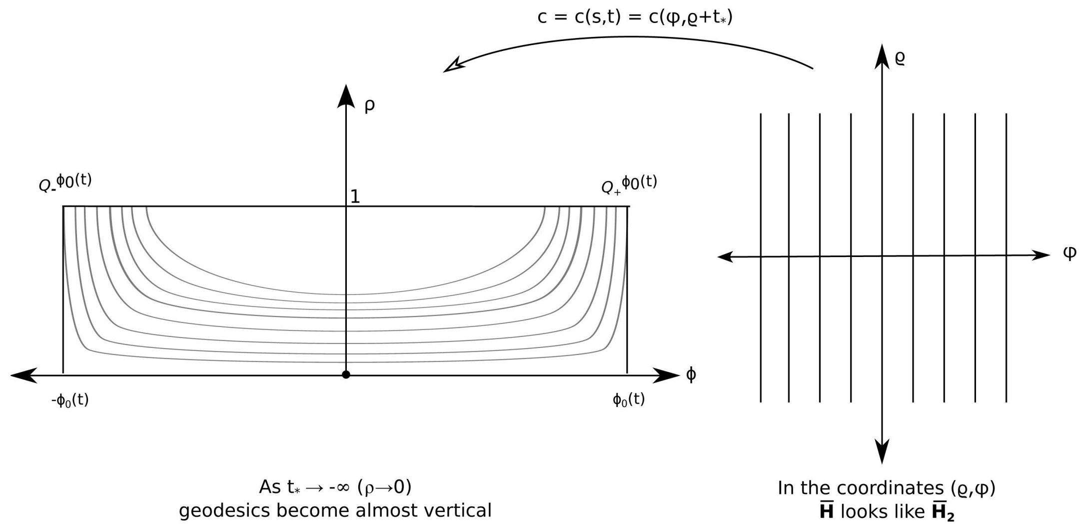
$$t \mapsto c_\rho(0, t) \text{ satisfies the equation } \frac{\partial c_\rho}{\partial t}(0, t) = c_\rho^3(0, t), \tag{12}$$

$$c_\rho(0, 1) = 1 \text{ and } c_\phi(0, t) = 0 \text{ for all } t \in \left(-\infty, \frac{3}{2}\right), \tag{13}$$

$$s \mapsto c(s, t) \text{ is a unit speed symmetric geodesic (cf. Definition 11).} \tag{14}$$

To motivate this construction, fix a parameter $t_* < 0$. As will be explicitly described in Section 5, we define coordinates (ϱ, φ) by

$$(\varrho, \varphi) \mapsto c(\varrho, \varphi + t_*).$$



In other words, the coordinates (ϱ, φ) not only depend on the family of geodesics $\{s \mapsto c(s,t)\}$ but also on the parameter t_*. We are interested in the asymptotics as $t_* \to -\infty$. More precisely, we want the metric expression of $g_{\mathbf{H}}$ with respect to (ϱ, φ) as $t_* \to -\infty$ to resemble to the metric expression of $g_{\mathbf{H}_2}$ with respect to the coordinates (ρ, ϕ) which is

$$g_{\mathbf{H}_2}(\rho, \phi) = \begin{pmatrix} 1 & 0 \\ 0 & \rho^6 \end{pmatrix}.$$

In other words, we want as $t_* \to -\infty$

$$g_{\mathbf{H}}(\varrho, \varphi) \approx \begin{pmatrix} 1 & 0 \\ 0 & \varrho^6 \end{pmatrix}.$$

The expression of the metric $g_{\mathbf{H}}$ in the coordinates (ϱ, φ) is

$$\begin{aligned} g_{\mathbf{H}}(\varrho, \varphi) &= \begin{pmatrix} |\frac{\partial c}{\partial s}(\varrho, \varphi + t_*)|^2 & \langle \frac{\partial c}{\partial s}(\varrho, \varphi + t_*), \frac{\partial c}{\partial t}(\varrho, \varphi + t_*)\rangle \\ \langle \frac{\partial c}{\partial s}(\varrho, \varphi + t_*), \frac{\partial c}{\partial t}(\varrho, \varphi + t_*)\rangle & |\frac{\partial c}{\partial t}(\varrho, \varphi + t_*)|^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & |\frac{\partial c}{\partial t}(\varrho, \varphi + t_*)|^2 \end{pmatrix}. \end{aligned}$$

The top diagonal entry is equal to 1 because of (14). The off-diagonal terms are equal to 0 because of the following reason: First, note that the curve $t \mapsto c(0,t)$ parametrizes the line $\phi = 0$ by (12) and (13). Next, since the geodesic $s \mapsto c(s,t)$ is symmetric (cf. (14)), the minimum value of the function $s \mapsto c_\rho(s,t)$ is achieved at $s = 0$ by Observation 1 (i). In particular, $\frac{\partial c_\rho}{\partial s}(0,t) = 0$ which in turn implies $\frac{\partial c}{\partial s}(0,t)$ is parallel to the line $\rho = 0$. Therefore, we conclude that the Jacobi field $\frac{\partial c}{\partial t}$ is perpendicular to the velocity vector $\frac{\partial c}{\partial s}$ of the geodesic at $s = 0$, and they must be perpendicular for all s by a standard property of Jacobi fields. This justifies that the off-diagonal entries are equal to 0.

The examination of the bottom diagonal entry is more subtle than that of the other entries and explains the role of the differential equation given in (12). First, note that (for fixed $t = \varphi + t_*$) the geodesic $s \mapsto c(s,t)$ can be written in the notation of Observation 1 as $\gamma^{\phi_0(t)}$, where $\phi_0(t) \to -\infty$ as $t \to -\infty$. Observation 1 implies that the geodesic $s \mapsto c_\rho(s,t)$ asymptotically converges to the piecewise geodesic $s \mapsto \sigma(s,t)$ made up of two geodesic curves $s \mapsto (-s, -\phi_0(t))$ for $s \in (-\infty, 0]$ and $s \mapsto (s, \phi_0(t))$ for $s \in [0, \infty)$ as $t \to -\infty$. In particular, this implies

$$|c_\rho(s,t) - |s|| = |c_\rho(s,t) - \sigma_\rho(s,t)| \to 0. \tag{15}$$

By invoking the well-known fact for a sequence of harmonic maps (in particular geodesics) local C^0-convergence implies a local C^k-convergence, we conclude that

$$\left|\frac{\partial c_\phi}{\partial t}\right| = \left|\frac{\partial c_\phi}{\partial t} - \frac{\partial \sigma_\phi}{\partial t}\right| \to 0 \quad \text{uniformly as } t \to -\infty$$

in any compact interval I contained in $(-\infty, 0) \cup (0, \infty)$. Thus, bottom diagonal entry is

$$\left|\frac{\partial c}{\partial t}(s,t)\right|^2 \approx \left(\frac{\partial c_\rho}{\partial s}(s,t)\right)^2$$

for $s \in I$ and $-t$ large. If we assume

$$\frac{\partial c_\rho}{\partial t}(s,t) = c_\rho^3(s,t), \tag{16}$$

then we can conclude $|\frac{\partial c}{\partial t}(s,t)|^2 \approx c_\rho^6(s,t)$ as $t \to -\infty$. Since $c_\rho^6(s,t) \approx s^6$ by (15), we therefore observe back in the coordinates (ϱ, φ)

$$g_{\mathbf{H}}(\varrho, \varphi) = \begin{pmatrix} 1 & 0 \\ 0 & |\frac{\partial c}{\partial t}(\varrho, \varphi + t_*)|^2 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & \varrho^6 \end{pmatrix} \quad \text{as } t_* \to -\infty. \tag{17}$$

On the other hand, imposing assumption (16) along with (14) results in an over-determined set of conditions for c. Thus, we impose (12) instead and we show that assumption (12) is sufficiently strong to prove the desired estimates for the coordinate expression of $g_{\mathbf{H}}$ with respect to (ϱ, φ). Note that in (17) above, we only gave evidence for C^0-estimates of the Jacobi field $s \mapsto \frac{\partial c}{\partial t}(s,t)$. However, in order to study harmonic maps we need higher derivative estimates of the metric. Obtaining higher derivative estimates for the asymptotic behavior of the Jacobi field $s \mapsto \frac{\partial c}{\partial t}(s,t)$ is much more subtle and constitutes the majority of Section 4. The discussion above is illustrated in the picture in the previous page.

In Section 5, we introduce the coordinate system (ϱ, φ) in $\mathbf{H}$ described above. We write as in (17)

$$g_{\mathbf{H}} = d\varrho^2 + \mathcal{J}^2(\varrho, \varphi) d\varphi^2, \tag{18}$$

where $\mathcal{J}(\varrho, \varphi) = |J(\rho, \phi)|$, $J(\rho, \phi) = \frac{\partial}{\partial \varphi}$ is the Jacobi field associated to the one-parameter family of geodesics $\{\varrho \mapsto (\varrho, \varphi)\}$ and we prove precise estimates for the metric and its Christoffel symbols.

Finally, in Section 6, we prove the existence of essentially regular subsets in $\overline{\mathbf{H}}$. The precise statement is contained in Theorem 28. Its proof is an adaptation of the arguments in Section 3 for the existence of essentially regular subsets in $\overline{\mathbf{H}}_2$ using the metric comparison estimates of Section 5.

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2. Preliminaries

Let Y denote either $\overline{\mathbf{H}}$ or $\overline{\mathbf{H}}_2$, h denote either $g_{\mathbf{H}}$ or $d_{\mathbf{H}_2}$ and d denote either $d_{\mathbf{H}}$ or $d_{\mathbf{H}_2}$ for simplicity. The *homogeneous coordinates* (ρ, Φ) of Y are defined by setting

$$\Phi = \rho^3 \phi.$$

It can be easily seen that the metric of Y is invariant under the scaling

$$\rho \to \lambda\rho, \quad \Phi \to \lambda\Phi.$$

Thus, the distance function of Y is homogeneous of degree 1 under this scaling. More precisely, for P given by (ρ, Φ) in homogeneous coordinates if $P \neq P_0$ and $\lambda \in (0, \infty)$, we denote by λP the point given by

$$(\lambda\rho, \lambda\Phi) \quad \text{and} \quad \lambda P_0 = P_0. \tag{19}$$

Then

$$d(\lambda P, \lambda Q) = \lambda d(P, Q).$$

We note that in the original coordinates (ρ, ϕ) of Y, we have

$$\lambda(\rho, \phi) = (\lambda\rho, \lambda^{-2}\phi). \tag{20}$$

Furthermore, if $\gamma(s)$ is an arclength parameterized geodesic, then $s \mapsto \lambda^{-1}\gamma(\lambda s)$ is also an arclength parameterized geodesic. Furthermore, an immediate computation yields the following:

Lemma 3. *In homogeneous coordinates (ρ, Φ), the metric of $\mathbf{H}$ is given by*

$$h = \begin{pmatrix} 1 + 9\Phi^2\rho^{-2} & -3\rho^{-1}\Phi \\ -3\rho^{-1}\Phi & 1 \end{pmatrix}$$

or equivalently

$$\begin{aligned} h &= (1 + 9\Phi^2\rho^{-2})d\rho^2 - 3\rho^{-1}\Phi d\rho d\Phi - 3\rho^{-1}\Phi d\Phi d\rho + d\Phi^2 \\ &= d\rho^2 + (3\Phi\rho^{-1}d\rho - d\Phi)^2. \end{aligned}$$

Lemma 4. *For $P_1 = (\rho_1, \phi_1)$, $P_2 = (\rho_2, \phi_2) \in Y$, we have*

$$|\rho_1 - \rho_2| \le d(P_1, P_2) \le |\rho_1 - \rho_2| + |\Phi_1 - \Phi_2|$$

where $\Phi_1 = \rho_1^3\phi_1$ and $\Phi_2 = \rho_2^3\phi_2$.

Proof. If P_1 and P_2 lie in opposite copies of $\mathbf{H}$, then we have $d(P_1, P_2) = |\rho_1 - \rho_2|$ (see our sign convention for ρ in the remarks preceding (4)) and thus the lemma holds. So, let $\gamma = (\gamma_\rho, \gamma_\phi) : [0, 1] \to \mathbf{H}$ be a geodesic from P_1 to P_2. Then

$$|\rho_1 - \rho_2| \le \int_0^1 |\gamma_\rho'|\, ds \le \int_0^1 |\gamma'|_h\, ds = d(P_1, P_2).$$

Next, note that since $(\rho, \phi) \mapsto (\rho, \phi - \phi_1)$ is an isometry, we can assume without the loss of generality that $\phi_1 = 0$. Along the line $\Phi = \rho^3\phi = 0$, the metric on $\mathbf{H}$ is given in coordinates (ρ, Φ) by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, setting $\alpha(t) = ((1-t)\rho_1 + t\rho_2, 0)$, we have

$$\int_0^1 |\alpha'_\rho|_h \, dt = |\rho_1 - \rho_2|$$

Furthermore, setting $\beta(t) = (\rho_2, t\Phi_2)$, we have

$$|\beta'_\Phi|_h = |\Phi_2|.$$

The join of the curves $\alpha(t)$ with $0 \le t \le 1$ and $\beta(t)$ with $0 \le t \le 1$ connects P_1 to P_2. Thus, we obtain

$$\begin{aligned} d(P_1, P_2) &\le \int_0^1 |\alpha'_\rho|_h \, dt + \int_0^1 |\beta'_\Phi|_h \, dt \\ &= |\rho_1 - \rho_2| + |\Phi_2|. \end{aligned}$$

□

For a map $v : (\Omega, g) \to Y$ from a bounded Riemannian domain, let the function $|\nabla v|^2$ be the energy density as defined in [15]. The *energy* of v is

$$E^v = \int_\Omega |\nabla v|^2 \, d\mu.$$

Definition 5. The map $u : \Omega \to Y$ is said to be *harmonic* if for every $x \in \Omega$, there exists $r > 0$ such that $u\big|_{B_r(x)}$ is energy minimizing with respect to all finite energy maps $v : B_r(x) \to Y$ with the same trace (cf. [15]).

Harmonic maps $u : (\Omega, g) \to Y$ have the following important monotonicity formula. Given $x_0 \in \Omega$ and $\sigma > 0$ such that $B_\sigma(x_0) \subset \Omega$, identify $x_0 = 0$ via normal coordinates and let

$$E^u(\sigma) := \int_{B_\sigma(0)} |\nabla u|^2 \, d\mu \quad \text{and} \quad I^u(\sigma) := \int_{\partial B_\sigma(0)} d^2(u, u(x)) \, d\Sigma.$$

There exists a constant $c > 0$ depending only on the C^2 norm of the metric on g (with $c = 0$ when g is the standard Euclidean metric) such that

$$\sigma \mapsto e^{c\sigma^2} \frac{\sigma E^u(\sigma)}{I^u(\sigma)}$$

is non-decreasing. As a non-increasing limit of continuous functions,

$$\mathrm{Ord}^u(x_0) := \lim_{\sigma \to 0} e^{c\sigma^2} \frac{\sigma E^u(\sigma)}{I^u(\sigma)}$$

is an upper semicontinuous function and $\mathrm{Ord}^u(x_0) \ge 1$. The value $\mathrm{Ord}^u(x_0)$ is called the order of u at x_0 (See [12, Section 1.2] with [15] and [16] to justify the various technical steps.)

Let $u : B_R(0) \to Y$ be a harmonic map. Recall from Section 1 that the singular set of u is defined as

$$\mathcal{S}(u) = \{x \in B_R(0) : u(x) = P_0\}.$$

A *singular point* is a point in $S(u)$ and a *regular point* is a point that is not a singular point. In a neighborhood of $x \in B_R(0)\backslash\mathcal{S}(u)$, u maps into a smooth Riemannian manifold $\mathbf{H}$, and we can write

$$u = (u_\rho, u_\phi)$$

in terms of coordinates (ρ, ϕ). The local Lipschitz continuity of u (cf. [15, Theorem 2.4.6]) implies that, for every $r \in (0, R)$, there exists a constant C dependent only on r and the total energy of u such that

$$|\nabla u_\rho| \leq C \quad \text{and} \quad |u_\rho^3 \nabla u_\phi| \leq C \quad \text{in } B_r(0)\backslash\mathcal{S}(u). \tag{21}$$

Furthermore, u satisfies the harmonic map equations (6) in $B_R(0)\backslash\mathcal{S}(u)$ and weakly in $B_R(0)$. In particular it follows from (6) and (21) that there is a constant C dependent only on r and the total energy of u such that

$$|\triangle u_\rho| \leq \frac{C}{u_\rho} \quad \text{and} \quad |\triangle u_\phi| \leq \frac{C}{u_\rho^4} \quad \text{in } B_r(0)\backslash\mathcal{S}(u). \tag{22}$$

An important tool used by Gromov and Schoen in [12] is the blow-up analysis of a harmonic map in the singular setting. The target spaces in [12] are non-positively curved Riemannian simplicial complexes. Though they differ from Riemannian manifolds in a number of ways, they have the nice feature that their tangent spaces are conical Euclidean simplicial complexes. After proving an important monotonicity formula for a harmonic map $u : X \to Y$, Gromov and Schoen establish, at each point $x \in X$, the existence of a sequence of blow-up maps u_{σ_i} converging to a tangent map $u_* : B_1(0) \to T_{u(x)}Y$ into a tangent space of Y. In the special case when Y is $\mathbb{R}$ (i.e. u is a harmonic function), u_* is the homogeneous harmonic polynomial approximating $u - u(x_0)$ near x.

In the present situation, we define blow-up maps in the similar way as [12] (cf. Definition 7 below), but the analysis of these blow-up maps is complicated by non-local compactness of the spaces $\overline{\mathbf{H}}$ or $\overline{\mathbf{H}}_2$ as indicated in the Introduction. Indeed, although we know the blow-up maps are uniformly continuous (in fact, the normalization of the blow-up maps implies a uniform bound on the energies and hence a uniform Lipschitz continuity in any compactly contained set in the domain by [15, Theorem 2.4.6]), the non-local compactness prohibits us from using the usual argument involving the Arzela–Ascoli Theorem. This is the point of invoking the generalized version of Arzela–Ascoli developed in [16, Section 3].

In doing so, we obtain a homogeneous map u_* *into an abstract NPC space* Y_* (cf. (26) below). Because we cannot compare u_* with a blow-up map $u_\sigma : B_1(0) \to \overline{\mathbf{H}}_2$ (since the target spaces are different for these maps as explained above), we need Lemma 9 where we view the homogeneous degree 1 maps $\{L_{\sigma_i}\}$ as the tangent maps embedded into $\overline{\mathbf{H}}_2$. The point of this lemma is that the blow-up maps $\{u_{\sigma_i}\}$ and the embedded tangent maps $\{L_{\sigma_i}\}$ (both mapping into the same space) become arbitrarily close as $\sigma_i \to 0$. We also need an analogous lemma for the target $\overline{\mathbf{H}}$. Although Lemma 9 and Lemma 10 are similar, we highlight an important difference: the image of L_{σ_i} in Lemma 10 is contained in a geodesic while the image of L_{σ_i} in Lemma 9 is not. In particular, this means that the former sequence of maps are harmonic maps while the latter sequence of maps are not. Thus, Lemma 10 is only a preliminary result; indeed, the correct analogy of Lemma 9 for the target $\overline{\mathbf{H}}$ is Lemma 12. The relationship between the preliminary Lemma 10 and Lemma 12 can be explained by Observation 1.

We now give the details of the blow-up construction described above. First, we need that the domain metric is expressed with respect to normal coordinates so we make the following definition.

Definition 6. A smooth Riemannian metric g on $B_R(0) \subset \mathbb{R}^n$ is said to be *normalized* if the standard Euclidean coordinates $(x^1, \ldots, x^n)$ are normal coordinates of g. The metric g_s for $s \in (0, R]$ on $B_1(0)$ is defined by

$$g_s(x) = g(sx).$$

Given a normalized metric g on $B_R(0)$ and a harmonic map $u : (B_R(0), g) \to Y$, the homogeneous coordinates can be used to define blow-up maps of u at 0. More precisely, we write

$$u = (u_\rho, u_\Phi)$$

in coordinates (ρ, Φ).

Definition 7. For $\sigma \in (0, R]$, define a harmonic map (which will be referred to as a *blow-up map*)

$$u_\sigma = (u_{\sigma\rho}, u_{\sigma\Phi}) : (B_1(0), g_\sigma) \to Y \tag{23}$$

by setting

$$u_{\sigma\rho}(x) = \mu^{-1}(\sigma) u_\rho(\sigma x) \quad \text{and} \quad u_{\sigma\Phi}(x) = \mu^{-1}(\sigma) u_\Phi(\sigma x)$$

or equivalently, writing $u_\sigma = (u_{\sigma\rho}, u_{\sigma\phi})$,

$$u_{\sigma\rho}(x) = \mu^{-1}(\sigma) u_\rho(\sigma x) \quad \text{and} \quad u_{\sigma\phi}(x) = \mu^{2}(\sigma) u_\phi(\sigma x)$$

where

$$\mu(\sigma) = \sqrt{\frac{I^u(\sigma)}{\sigma^{n-1}}}. \tag{24}$$

The choice of the scaling constant $\mu(\sigma)$ implies that

$$I^{u_\sigma}(1) = \int_{\partial B_1(0)} d^2(u_\sigma, P_0)\, d\Sigma = 1. \tag{25}$$

By the monotonicity property stated above, we have $E^{u_\sigma}(1) \leq 2\,\mathrm{Ord}^u(0)$ for $\sigma > 0$ sufficiently small. We can now use the Arzela–Ascoli-type construction in [16, Section 3]. First, by [15, Theorem 2.4.6], $\{u_\sigma\}$ has a uniform modulus of continuity. Therefore, by [16, Proposition 3.7], given a sequence u_{σ_i} with $\sigma_i \to 0$, there exists a subsequence and a map (referred to as a *tangent map*)

$$u_* : B_1(0) \to (Y_*, d_*) \tag{26}$$

into an NPC space such that

$$d(u_{\sigma_i}(\cdot), u_{\sigma_i}(\cdot)) \to d_*(u_*(\cdot), u_*(\cdot)) \quad \text{uniformly on compact sets.}$$

Following [16], we will refer to the convergence given above as *convergence in the pullback sense*. By using the same arguments as in [12], we can show that u_* is a homogeneous map of degree $\alpha = \mathrm{Ord}^u(0)$, i.e. $d(u_*(x), u_*(0)) = |x|^\alpha d(u_*(\frac{x}{|x|}, u(0))$ and the curve $t \mapsto u_*(tx)$ is a geodesic in Y_* for each $x \in \partial B_1(0)$.

The singular set $\mathcal{S}(u) = \{x \in B_R(0) : u(x) = P_0\}$ is partitioned into the two sets

$$\mathcal{S}_0(u) = \{x \in \mathcal{S}(u) : \mathrm{Ord}^u(x) > 1\}$$

and

$$\mathcal{S}_1(u) = \{x \in \mathcal{S}(u) : \mathrm{Ord}^u(x) = 1\}.$$

The following is Theorem 35 of the Appendix.

Theorem 8. *If $u : B_1(0) \to Y$ is a harmonic map, then the set of higher order points of u is of Hausdorff codimension at least 2, i.e.*

$$\dim_{\mathcal{H}}(\mathcal{S}_0(u)) \leq n - 2.$$

The qualitative behavior harmonic maps at order one points are given by the next two lemma which follow immediately from Lemma 36 of the Appendix.

Lemma 9. *Let g be a normalized metric on $B_1(0)$ and let $u : (B_1(0), g) \to (\overline{\mathbf{H}}_2, d)$ be a harmonic map with $\mathrm{Ord}^u(0) = 1$ and $u(0) = P_0$. Then given a sequence $\sigma_i \to 0$ there exists a subsequence (denoted again by σ_i) a rotation $R : \mathbb{R}^n \to \mathbb{R}^n$, a sequence of homogeneous degree 1 maps $L_{\sigma_i} : B_1(0) \to \overline{\mathbf{H}}_2$ defined by*

$$L_{\sigma_i}(x) = \begin{cases} (Ax^1, \phi^+_{\sigma_i}), & x^1 > 0, \\ P_0, & x^1 = 0, \\ (Ax^1, \phi^-_{\sigma_i}), & x^1 < 0, \end{cases} \tag{27}$$

for a constant $A > 0$ and sequences $\{\phi^+_{\sigma_i}\}$, $\{\phi^-_{\sigma_i}\}$ such that, for any $r \in (0, 1)$,

$$\lim_{i\to\infty} \sup_{B_r(0)} d(u_{\sigma_i} \circ R, L_{\sigma_i}) = 0,$$

where u_{σ_i} are the blow-up maps u at x.

Lemma 10. *Let g be a normalized metric on $B_1(0)$ and let $u : (B_1(0), g) \to (\overline{\mathbf{H}}, d)$ be a harmonic map with $\mathrm{Ord}^u(0) = 1$ and $u(0) = P_0$. Then given a sequence $\sigma_i \to 0$ there exists a subsequence (denoted again by σ_i), a rotation $R : \mathbb{R}^n \to \mathbb{R}^n$, a sequence of homogeneous degree 1 maps $L_{\sigma_i} : B_1(0) \to \overline{\mathbf{H}}$ defined by*

$$L_{\sigma_i}(x) = \begin{cases} (Ax^1, \phi^+_{\sigma_i}), & x^1 > 0, \\ P_0, & x^1 = 0, \\ (-Ax^1, \phi^-_{\sigma_i}), & x^1 < 0, \end{cases} \tag{28}$$

for a constant $A > 0$ and sequences $\{\phi^+_{\sigma_i}\}$, $\{\phi^-_{\sigma_i}\}$ such that, for any $r \in (0, 1)$,

$$\lim_{i\to\infty} \sup_{B_r(0)} d(u_{\sigma_i} \circ R, L_{\sigma_i}) = 0$$

where u_{σ_i} are the blow-up maps u at x_0.

As mentioned above, the image of L_{σ_i} in Lemma 10 (unlike that the image of L_{σ_i} in Lemma 9) is not contained in a geodesic. We will need to work a little harder to obtain a lemma analogous to Lemma 9 for target $\overline{\mathbf{H}}$. By composing with a rotation if necessary, we may assume that $\{u_{\sigma_i}\}$ converges in the pullback sense to the linear function

$$u_*(x) = Ax^1 \tag{29}$$

for some constant $A > 0$. Furthermore, let

$$c_i = \frac{u_{\sigma_i\phi}(1) + u_{\sigma_i\phi}(-1)}{2}$$

and define an isometry $T_{c_i} : \overline{\mathbf{H}} \to \overline{\mathbf{H}}$ by setting

$$T_{c_i}(P_0) = P_0 \quad \text{and} \quad T_{c_i}(\rho, \phi) = (\rho, \phi - c_i).$$

Since $(T_{c_i} \circ u_{\sigma_i})_\phi(1) = (T_{c_i} \circ u_{\sigma_i})_\phi(-1)$, we can assume that the sequence $\{u_{\sigma_i}\}$ satisfy

$$u_{\sigma_i\phi}(1) = u_{\sigma_i\phi}(-1), \quad i = 1, 2, \ldots. \tag{30}$$

Below, we will assume that u_* and u_{σ_i} satisfy the normalization assumptions (29) and (30).

Definition 11. An arclength parameterized geodesic

$$\gamma = (\gamma_\rho, \gamma_\phi) : (-\infty, \infty) \to \mathbf{H}$$

is said to be *symmetric* if

$$\gamma_\rho(s) = \gamma_\rho(-s) \quad \text{and} \quad \gamma_\phi(s) = -\gamma_\phi(-s).$$

A homogeneous degree 1 map

$$l : B_1(0) \to \mathbf{H}$$

is said to be a *symmetric homogeneous degree* 1 *map* if

$$l(x) = \gamma(Ax)$$

for some $A > 0$ and some symmetric geodesic. We call the number $\gamma_\phi(1)$ the *address* and A the *stretch* of a symmetric of l.

As mentioned above, Lemma 10 should only be considered a preliminary result since L_{σ_i} is not a harmonic map. To proceed to our final goal (cf. Lemma 12 below), we make the observation that there exists a geodesic whose image is close to the image of the map L_{σ_i}. Claim (5) of Observation 1 leads us to the following modification of Lemma 10.

Lemma 12. *Let g be a normalized metric on $B_1(0)$, $u : B_1(0) \to (\overline{\mathbf{H}}, d)$ a harmonic map with* $\mathrm{Ord}^u(0) = 1$, *$u(0) = P_0$ and let $\{u_{\sigma_i}\}$ be as in Lemma* 10 *normalized such that* (29) *and* (30) *are satisfied. Then there exists a sequence of symmetric geodesics γ_{σ_i} with the address $\gamma_{\sigma_i\phi}(1) \to \infty$, a sequence of symmetric homogeneous degree* 1 *maps $l_{\sigma_i} : B_1(0) \to \overline{\mathbf{H}}$ given by*

$$l_{\sigma_i}(x) = \gamma_{\sigma_i}(Ax^1)$$

and a subsequence (denoted again by σ_i) such that

$$\lim_{i\to\infty} \sup_{B_r(0)} d(u_{\sigma_i}, l_{\sigma_i}) = 0$$

for any $r \in (0, R)$.

Proof. Let $\phi^{\pm}_{\sigma_i}$ be as in (28). We claim that $\phi^{+}_{\sigma_i} \to \infty$. Indeed, if this claim is not true, then (by taking a subsequence if necessary) $\phi^{+}_{\sigma_i} \to \phi_\infty$. By the normalization (30), we can have $\phi^{-}_{\sigma_i} \to -\phi_\infty$. Thus, L_{σ_i} converges uniformly to a homogeneous degree 1 map

$$L_\infty(x) = \begin{cases} (Ax^1, \phi_\infty), & x^1 > 0, \\ P_0, & x^1 = 0, \\ (Ax^1, -\phi_\infty), & x^1 < 0, \end{cases}$$

and

$$\lim_{i\to\infty} \sup_{B_r(0)} d(u_{\sigma_i}, L_\infty \circ R) = 0.$$

Since u_{σ_i} is a harmonic map, so is L_∞. By the maximum principle,

$$d(L_\infty(0), P) \neq \sup_{B_1(0)} d(L_\infty, P).$$

Since we have $L_\infty(0) = P_0$ and $\sup_{B_1(0)} d(L_\infty, P) = d(P_0, P)$ for P far P away from the image of L_∞, this is a contradiction. Set γ_{σ_i} to be the symmetric geodesic and with address $\gamma_{\sigma_i\rho}(1) = \phi_{\sigma_i}$. Since $\phi_{\sigma_i} \to \infty$, we have that (cf. Observation 1)

$$\lim_{i\to\infty} \sup_{B_1(0)} d(l_{\sigma_i}, L_{\sigma_i}) = 0.$$

Thus,

$$\lim_{i\to\infty} \sup_{B_r(0)} d(u_{\sigma_i}, l_{\sigma_i}) = \lim_{i\to\infty} \sup_{B_r(0)} d(u_{\sigma_i}, L_{\sigma_i}). \qquad \square$$

3. Essential regularity in $\overline{\mathbf{H}}_2$

The goal of this section is to show that the space $\overline{\mathbf{H}}_2[\phi_0] = \{(\rho,\phi) \in \overline{\mathbf{H}}_2 : |\phi| \leq \phi_0\}$ satisfies a property which is similar (but not equivalent) to being *essentially regular* in the sense of [12, Section 5]. For a harmonic map $v : \Omega \to \overline{\mathbf{H}}_2$ from a Riemannian domain, write $v = (v_\Upsilon, v_\Phi)$ with respect to coordinates $\Upsilon := \rho - \frac{3}{2}\rho^5\phi^2$ and $\Phi := \rho^3\phi$. Thus,

$$v_\Upsilon = v_\rho - \frac{3}{2} v_\rho^5 v_\phi^2 \quad \text{and} \quad v_\Phi = v_\rho^3 v_\phi. \tag{31}$$

In $\Omega \backslash \mathcal{S}(v)$, we have the harmonic map equation

$$\triangle v_\Upsilon = -\frac{45}{2} v_\rho^9 v_\phi^2 |\nabla v_\phi|^2 - 30 v_\rho^3 v_\phi^2 |\nabla v_\rho|^2 - 12 v_\rho^4 v_\phi \nabla v_\rho \cdot \nabla v_\phi \tag{32}$$

and

$$\triangle v_\Phi = 9 v_\rho^7 v_\phi |\nabla v_\phi|^2 + 6 v_\rho v_\phi |\nabla v_\rho|^2. \tag{33}$$

We will denote a geodesic ball of radius R centered at P_0 in $\overline{\mathbf{H}}_2$ by $\mathbf{B}_R(P_0)$.

Lemma 13. *Let $R \in [\frac{1}{2}, 1)$, $E_0 > 0$, $A_0 > 0$ and a normalized metric g on $B_R(0)$ be given. Then there exists $C \geq 1$ depending only on E_0, A_0 and g such that if $\phi_0 > 0$, $s \in (0,1]$, $\vartheta \in (0,1]$ and*

$$v : (B_{\vartheta R}(0), g_s) \to \overline{\mathbf{H}}_2\left[\frac{\phi_0}{\vartheta^2}\right] \cap \mathbf{B}_{A_0\vartheta}(P_0)$$

is a harmonic map with

$$E^v \leq \vartheta^n E_0,$$

then

$$|v_\rho^3 \nabla v_\phi|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} \leq C\phi_0,$$

and

$$|\triangle v_\Upsilon|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} \leq C\vartheta^{-1}\phi_0^2.$$

Proof. Throughout the proof, C will denote a generic constant dependent on only E_0, A_0 and g. The assumption on the bounds of $|v_\rho|$ and $|v_\phi|$ imply that

$$|v_\Phi|_{L^\infty(B_{\vartheta R}(0))} = |v_\rho^3 v_\phi|_{L^\infty(B_{\vartheta R}(0))} \leq C\vartheta\phi_0.$$

Furthermore, (21) and (33) imply

$$|\triangle v_\Phi|_{L^\infty(B_{\vartheta R}(0))} \leq C\vartheta^{-1}\phi_0.$$

Next, note that v_Υ and v_Φ satisfy the equalities (32) and (33) weakly in $B_R(0)$. Thus, by elliptic regularity, for any $\alpha \in (0, 1)$

$$\vartheta|\nabla v_\Phi|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} \leq C(\vartheta^2|\triangle v_\Phi|_{L^\infty(B_{\vartheta R}(0))} + |v_\Phi|_{L^\infty(B_{\vartheta R}(0))}) \leq C\vartheta\phi_0. \tag{34}$$

Since $\nabla v_\Phi = \nabla(v_\rho^3 v_\phi) = v_\rho^3 \nabla v_\phi + 3v_\rho^2 v_\phi \nabla v_\rho$, we have

$$|v_\rho^3 \nabla v_\phi| \leq 3|v_\rho^2 v_\phi \nabla v_\rho| + |\nabla v_\Phi|.$$

The assumption on the bounds of $|v_\rho|$ and $|v_\phi|$ and (21) imply that the first term on the right-hand side above is bounded in $B_{\frac{15\vartheta R}{16}}(0)$ by $C\phi_0$. By (34), the second term is also bounded by $C\phi_0$. Thus, we obtain the first estimate of the lemma. Combining the first estimate, the assumption on the bounds of $|v_\rho|$ and $|v_\phi|$, (21) and (32), we obtain the second estimate. □

Definition 14. We say that a map $l = (l_\rho, l_\phi) : B_1(0) \to \overline{\mathbf{H}}_2$ is an *almost affine map* if $l_\rho(x) = \vec{a} \cdot x + b$ for $\vec{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, i.e. l_ρ is an affine function.

We are now ready to prove the main theorem of the section.

Theorem 15. *Let $R \in [\frac{1}{2}, 1)$, $E_0 > 0$, $A_0 > 0$ and a normalized metric g on $B_R(0)$ be given. There exist $C \geq 1$ and $\alpha > 0$ depending only on E_0, A_0 and g with the following property: For $\phi_0 > 0$, $s \in (0, 1]$, $\vartheta \in (0, 1]$, if*

$$v : (B_{\vartheta R}(0), g_s) \to \overline{\mathbf{H}}_2\left[\frac{\phi_0}{\vartheta^2}\right] \cap \mathbf{B}_{A_0\vartheta}(P_0)$$

is a harmonic map with

$$E^v \leq \vartheta^n E_0,$$

then

$$\sup_{B_{r\vartheta}(0)} d(v, \hat{l}) \leq Cr^{1+\alpha} \sup_{B_{\vartheta R}(0)} d(v, L) + Cr\vartheta\phi_0^2 \quad \textit{for all } r \in \left(0, \frac{R}{2}\right] \tag{35}$$

where $\hat{l} = (\hat{l}_\rho, \hat{l}_\phi) : B_1(0) \to \overline{\mathbf{H}}_2$ *is the almost affine map given by*

$$\hat{l}_\rho(x) = v_\rho(0) + \nabla v_\rho(0) \cdot x, \quad \hat{l}_\phi(x) = v_\phi(x)$$

and $L : B_1(0) \to \overline{\mathbf{H}}_2$ *is any almost affine map.*

Remark 16. A subset $E \subset \overline{\mathbf{H}}_2$ will be called *essentially regular* if any harmonic map $v : \Omega \to E$ from a Riemannian domain satisfies (35) above.

Proof. Throughout the proof, let C be a generic constant that depends on E_0, A_0 and g. Since

$$v_\Upsilon - \hat{l}_\rho = v_\rho - \hat{l}_\rho - \frac{3}{2} v_\rho^5 v_\phi^2,$$

we have

$$\big| |v_\Upsilon - \hat{l}_\rho|_{L^\infty(B_{R\vartheta}(0))} - |v_\rho - \hat{l}_\rho|_{L^\infty(B_{R\vartheta}(0))} \big| \le C \vartheta^5 \left(\frac{\phi_0}{\vartheta^2} \right)^2 = C \vartheta \phi_0^2.$$

Thus, elliptic regularity and the second estimate of Lemma 13 imply that

$$\begin{aligned}\vartheta^{1+\alpha} [\nabla(v_\Upsilon - \hat{l}_\rho)]_{C^\alpha(B_{\frac{7\vartheta R}{8}}(0))} &\le C(\vartheta^2 |\triangle(v_\Upsilon - \hat{l}_\rho)|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} + |v_\Upsilon - \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))}) \\ &\le C \vartheta \phi_0^2 + C |v_\rho - \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))}.\end{aligned}$$

Since $\nabla v_\rho(0) = \nabla \hat{l}_\rho(0)$, we have

$$\begin{aligned}\sup_{B_{r\vartheta}(0)} |\nabla(v_\Upsilon - \hat{l}_\rho)| &\le C(r\vartheta)^\alpha \big(\vartheta^{-\alpha} \phi_0^2 + \vartheta^{-1-\alpha} |v_\rho - \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} \big) \\ &\le C r^\alpha \big(\phi_0^2 + \vartheta^{-1} |v_\rho - \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} \big).\end{aligned} \tag{36}$$

Furthermore, since

$$\nabla(v_\Upsilon - \hat{l}_\rho) = \nabla(v_\rho - \hat{l}_\rho) - 3 v_\rho^5 v_\phi \nabla v_\phi - \frac{15}{2} v_\rho^4 v_\phi^2 \nabla v_\rho,$$

the first estimate of Lemma 13 and (21) imply

$$\big| |\nabla v_\rho - \nabla \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} - |\nabla v_\Upsilon - \nabla \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} \big| \le C \phi_0^2. \tag{37}$$

Combined with the mean value inequality, we therefore conclude that for $r \in (0, \frac{R}{2}]$,

$$\begin{aligned}\sup_{B_{r\vartheta}(0)} |v_\rho - \hat{l}_\rho| &\le r\vartheta \sup_{B_{r\vartheta}(0)} |\nabla(v_\rho - \hat{l}_\rho)| \\ &\overset{(37)}{\le} r\vartheta \Big(\sup_{B_{r\vartheta}(0)} |\nabla(v_\Upsilon - \hat{l}_\rho)| + C\phi_0^2 \Big) \\ &\overset{(36)}{\le} r\vartheta \Big(C r^\alpha \big(\phi_0^2 + \vartheta^{-1} |v_\rho - \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} \big) + C\phi_0^2 \Big) \\ &\le C r^{1+\alpha} \sup_{B_{R\vartheta}(0)} |v_\rho - \hat{l}_\rho| + C r^{1+\alpha} \vartheta \phi_0^2 + C r \vartheta \phi_0^2 \\ &\le C r^{1+\alpha} \sup_{B_{R\vartheta}(0)} |v_\rho - \hat{l}_\rho| + C r \vartheta \phi_0^2.\end{aligned} \tag{38}$$

We claim

$$\sup_{B_{\vartheta R}(0)} |v_\rho - \hat{l}_\rho| \le C\Big(\sup_{B_{\vartheta R}(0)} |v_\rho - L_\rho| + \vartheta\phi_0^2\Big). \tag{39}$$

Assuming (39) is true, the above two inequalities imply

$$\sup_{B_{\vartheta r}(0)} |v_\rho - \hat{l}_\rho| \le Cr^{1+\alpha} \sup_{B_{\vartheta R}(0)} |v_\rho - L_\rho| + Cr\vartheta\phi_0^2 \quad \text{for all } r \in \Big(0, \frac{R}{2}\Big].$$

Since $v_\phi(x) = \hat{l}_\phi(x)$, this estimate and Lemma 4 imply the lemma.

We are left to prove (39). First, since $\hat{l}_\rho$ and L_ρ are affine functions and $\hat{l}_\rho(0) = v_\rho(0)$, we have for any $s \in (0, \frac{R}{2}]$,

$$\begin{aligned}
\sup_{B_{\vartheta R}(0)} |\hat{l}_\rho - L_\rho| &\le \sup_{B_{\vartheta R}(0)} \big(|(\hat{l}_\rho - L_\rho) - (\hat{l}_\rho(0) - L_\rho(0))| + |\hat{l}_\rho(0) - L_\rho(0)|\big) \\
&= s^{-1}R \sup_{B_{\vartheta s}(0)} \big(|(\hat{l}_\rho - L_\rho) - (\hat{l}_\rho(0) - L_\rho(0))|\big) + |\hat{l}_\rho(0) - L_\rho(0)| \\
&\le s^{-1}R \sup_{B_{\vartheta s}(0)} \big(|(\hat{l}_\rho - L_\rho) - (v_\rho(0) - L_\rho(0))| + |v_\rho(0) - L_\rho(0)|\big) \\
&\le s^{-1}R \sup_{B_{\vartheta s}(0)} |\hat{l}_\rho - L_\rho| + 2s^{-1}R \sup_{B_{\vartheta R}(0)} |v_\rho - L_\rho|.
\end{aligned}$$

Apply the triangle inequality and (38) to estimate the first term above by

$$\begin{aligned}
s^{-1}R \sup_{B_{\vartheta s}(0)} |\hat{l}_\rho - L_\rho| &\le s^{-1}R\Big(\sup_{B_{\vartheta s}(0)} |v_\rho - L_\rho| + \sup_{B_{\vartheta s}(0)} |v_\rho - \hat{l}_\rho|\Big) \\
&\le s^{-1}R \sup_{B_{\vartheta R}(0)} |v_\rho - L_\rho| + CRs^\alpha \sup_{B_{\vartheta R}(0)} |v_\rho - \hat{l}_\rho| + CR\vartheta\phi_0^2.
\end{aligned}$$

Combining the above two inequalities and noting $R \le 1$, we obtain

$$\sup_{B_{\vartheta R}(0)} |\hat{l}_\rho - L_\rho| \le 3s^{-1} \sup_{B_{\vartheta R}(0)} |v_\rho - L_\rho| + Cs^\alpha \sup_{B_{\vartheta R}(0)} |v_\rho - \hat{l}_\rho| + C\vartheta\phi_0^2. \tag{40}$$

Triangle inequality along with (40) imply

$$\begin{aligned}
\sup_{B_{\vartheta R}(0)} |v_\rho - \hat{l}_\rho| &\le \sup_{B_{\vartheta R}(0)} |v_\rho - L_\rho| + \sup_{B_{\vartheta R}(0)} |\hat{l}_\rho - L_\rho| \\
&\le (1 + 3s^{-1}) \sup_{B_{\vartheta R}(0)} |v_\rho - L_\rho| + Cs^\alpha \sup_{B_{\vartheta R}(0)} |v_\rho - \hat{l}_\rho| + C\vartheta\phi_0^2.
\end{aligned}$$

Choose $s \le \frac{R}{2}$ such that $Cs^\alpha \le \frac{1}{2}$ to obtain (39). □

4. Foliation of H by geodesics

In this section, we introduce a foliation of **H** by a family of geodesics

$$\{s \mapsto (c_\rho(s,t), c_\phi(s,t))\}.$$

As explained in the introduction, we note that an important feature of the coordinates (ρ, ϕ) of $\overline{\mathbf{H}}_2$ is that the curves $\{\rho \mapsto (\rho, \phi)\}$ define a family of geodesics. In order to replicate the essential regularity results of $\overline{\mathbf{H}}_2$ (cf. Section 3), we need a similar construction for $\overline{\mathbf{H}}$. We will use the map $\{(s,t) \mapsto (c_\rho(s,t), c_\phi(s,t))\}$ to define new coordinates (ϱ, φ) on $\overline{\mathbf{H}}$ resembling the coordinates (ρ, ϕ) of $\overline{\mathbf{H}}_2$. These new coordinates in turn will allow us to prove in Section 6 the essential regularity of $\overline{\mathbf{H}}$.

Consider the one parameter family of geodesics defined in the introduction (cf. (11)). By solving the differential equation (12) with initial condition (13), we obtain

$$c(0,t) = (c_\rho(0,t), c_\phi(0,t)) = ((3-2t)^{-1/2}, 0). \tag{41}$$

Since

$$\left(\frac{\partial c_\rho}{\partial s}\right)^2 + c_\rho^6 \left(\frac{\partial c_\phi}{\partial s}\right)^2 = 1 \quad \text{by (14)} \tag{42}$$

and

$$\frac{\partial c_\rho}{\partial s}(0,t) = 0 \quad \text{by (41)},$$

we conclude that $s \mapsto c(s,t)$ is the unique geodesic with initial conditions (41) and

$$\frac{\partial c}{\partial s}(0,t) = (0, (3-2t)^{3/2}).$$

For each $t \in (-\infty, \frac{3}{2})$, consider the vector field

$$s \mapsto X_t(s) := \frac{\partial c}{\partial t}(s,t) \quad \text{along } s \mapsto c(s,t)$$

and set

$$J_t(s) = J(s,t) = |X_t(s)|. \tag{43}$$

In particular, (41) implies

$$X_t(0) = \frac{\partial c}{\partial t}(0,t) = ((3-2t)^{-3/2}, 0)$$

and

$$J_t(0) = |X_t(0)| = (3-2t)^{-3/2} = c_\rho^3(0,t). \tag{44}$$

The symmetry of $s \mapsto c(s,t)$ implies the symmetry of J_t (i.e. $J_t(s) = J_t(-s)$) which in turn implies

$$J_t'(0) = 0. \tag{45}$$

Since X_t is generated by a family of geodesics, X_t is a Jacobi field and satisfies the differential equations

$$X_t''(s) + K_t(s) X_t(s) = 0, \tag{46}$$

where

$$K_t(s) = K(s,t) = -\frac{6}{c_\rho^2(s,t)} \tag{47}$$

is the Gauss curvature of $\mathbf{H}$ at $c(s,t)$. Since

$$\left\langle X_t(0), \frac{\partial c}{\partial s}(0,t) \right\rangle = 0,$$

the vector field X_t is orthogonal to the geodesic $s \mapsto c(s,t)$ at $s = 0$. Hence it is orthogonal for all s, i.e.

$$\left\langle X_t(s), \frac{\partial c}{\partial s}(s,t) \right\rangle = \left\langle \frac{\partial c}{\partial t}(s,t), \frac{\partial c}{\partial s}(s,t) \right\rangle = 0. \tag{48}$$

Furthermore,

$$\left\langle X_t'(s), \frac{\partial c}{\partial s}(s,t) \right\rangle = \left\langle \frac{\partial^2 c}{\partial s \partial t}(s,t), \frac{\partial c}{\partial s}(s,t) \right\rangle = \frac{1}{2} \frac{\partial}{\partial t} \left| \frac{\partial c}{\partial s}(s,t) \right|^2 = 0. \tag{49}$$

Combining (48) and (49), we conclude that both $X_t'(s)$ and $X_t(s)$ are perpendicular to $\frac{\partial c}{\partial s}$. By the two-dimensionality of $\mathbf{H}$, $X_t'(s)$ and $X_t(s)$ are parallel, and hence

$$|X_t'(s)|^2 |X_t(s)|^2 = \langle X_t'(s), X_t(s) \rangle^2.$$

Equation (46) implies

$$J_t''(s) = -K(s,t) J_t(s) + J_t(s)^{-3} \big(|X_t'(s)|^2 |X_t(s)|^2 - \langle X_t'(s), X_t(s) \rangle^2 \big).$$

The above two equalities along with (44) and (45) give us the differential equation along with boundary conditions

$$J_t'' + K J_t = 0, \quad J_t(0) = c_\rho^3(0,t), \quad J_t'(0) = 0. \tag{50}$$

Lemma 17. *For $c(s,t)$ given by* (11)*,*

$$|s| \le c_\rho(s,t).$$

Proof. Let $\gamma : (-\infty, \infty) \to \mathbf{H}$ be the unit speed symmetric geodesic satisfying

$$\gamma(0) = (1,0) \quad \text{and} \quad \gamma'(0) = (0,1). \tag{51}$$

We first claim that

$$\sigma < \gamma_\rho(\sigma) \quad \text{for all } \sigma \in [0, \infty). \tag{52}$$

Indeed, for a fixed $\sigma \in [0, \infty)$, consider the function

$$f : (0, \infty) \to \mathbb{R}^+, \quad f(\rho) = d((\rho, 0), \gamma(\sigma)).$$

Since $\rho \mapsto (\rho, 0)$ is a geodesic, $f(\rho)$ is a strictly convex function. Furthermore, the geodesic γ intersects the line $\{\phi = 0\}$ perpendicularly at $(1,0)$ by (51). Thus, $\rho = 1$ is the unique minimum of the function $f(\rho)$. In particular,

$$\sigma = d(\gamma(\sigma), (1,0)) = f(1) < \lim_{\rho \to 0} f(\rho) = d(P_0, \gamma(\sigma)) = \gamma_\rho(\sigma).$$

This proves (52). For each $t \in (0,\infty)$, we rescale the curve $s \mapsto c^t(s) := c(s,t)$, $s \in [-1,1]$ to define the unit speed geodesic (see remarks after (20))

$$\tilde{c}^t(\sigma) : [0, c_\rho^t(0)^{-1}] \to \mathbf{H}, \quad \tilde{c}^t(\sigma) := c_\rho^t(0)^{-1} c^t(c_\rho^t(0)\sigma).$$

Since $\tilde{c}^t(0) = (1,0)$ and $\frac{\partial \tilde{c}^t}{\partial s}(0) = (0,1)$, we conclude by (51) that $\tilde{c}^t(\sigma) = \gamma(\sigma)$. Consequently,

$$\sigma \le \tilde{c}_\rho^t(\sigma) \quad \text{for all } \sigma \in (0, c_\rho^t(0)^{-1}].$$

Multiply through by $a_t = c_\rho^t(0)$ to obtain

$$a_t\sigma \le c_\rho^t(a_t\sigma) \quad \text{for all } \sigma \in (0, c_\rho^t(0)^{-1}].$$

Letting $s = a_t\sigma$, we conclude

$$s \le c_\rho^t(s) \quad \text{for all } s \in [0,1]. \qquad \square$$

Lemma 18. *For $c(s,t)$ given by* (11) *and $J(s,t) = |\frac{\partial c}{\partial t}(s,t)|$, there exists $C > 1$ such that*

$$\begin{aligned} 0 \le \frac{\partial}{\partial s} \log J(s,t) &\le \frac{3}{c_\rho(s,t)} && \textit{for all } s \in [0,1],\\ -\frac{3}{c_\rho(s,t)} \le \frac{\partial}{\partial s} \log J(s,t) &\le 0 && \textit{for all } s \in [-1,0],\\ J(1,t)c_\rho^3(s,t) \le J(s,t) &\le c_\rho^3(s,t) && \textit{for all } s \in [-1,1]. \end{aligned}$$

Proof. Fix $t \in (0,\infty)$ and write $J(s) = J(s,t)$, $K(s) = K(s,t)$ and $a = a_t$ for simplicity. With this notation, we rewrite (50) as

$$J'' + KJ = 0, \quad J(0) = c_\rho^3(0,t), \quad J'(0) = 0. \tag{53}$$

Let

$$j(s) = c_\rho^3(s,t).$$

Combining the geodesic equation

$$c_\rho'' = 3c_\rho^5(c_\phi')^2$$

with (42), we obtain

$$j'' + Kj = 3c_\rho^2 c_\rho'' + 6c_\rho((c_\rho')^2 - 1) = 9c_\rho^7(c_\phi')^2 - 6c_\rho^7(c_\phi')^2 = 3c_\rho^7(c_\phi')^2.$$

Thus,

$$(J'(s)j(s) - J(s)j'(s))' = -3c_\rho^7(c_\phi')^2 J(s) \le 0.$$

Since

$$j'(0)J(0) - j(0)J'(0) = 0,$$

integration implies

$$J'(s)j(s) - J(s)j'(s) \le 0 \quad \text{for all } s \in [0,1].$$

Thus, we obtain

$$\frac{J'}{J} \le \frac{j'}{j} = \frac{3c_\rho'}{c_\rho} \quad \text{for all } s \in [0,1].$$

The first inequality of the lemma for $s \in [0,1]$ follows from noting that $|c_\rho'| \le |c'| = 1$. By symmetry, the second inequality follows from the same argument.

Using the initial condition $\frac{J(0)}{j(0)} = 1$, integration on the interval $[0, s]$ yields

$$J(s) \le j(s) = c_\rho^3 \quad \text{for all } s \in [0,1].$$

Since $j(1) = c_\rho^3(1,t) \ge 1$ by Lemma 17, integration on the interval $[s, 1]$ yields

$$J(1)c_\rho^3 \le \frac{J(1)}{j(1)} j(s) \le J(s) \quad \text{for all } s \in [0,1].$$

By symmetry, the same argument applies for $s \in [-1, 0]$ and completes the proof of the third inequality. □

Lemma 19. *For $c(s,t)$ given by* (11) *and $J(s,t) = |\frac{\partial c}{\partial t}(s,t)|$, there exists $C > 1$ such that for $(s,t) \in [-1,1] \times (-\infty, \frac{3}{2})$, we have*

$$\begin{aligned} \left|\frac{\partial J}{\partial s}(s,t)\right| &\le C c_\rho^{-1}(s,t) J(s,t), \\ \left|\frac{\partial^2 J}{\partial s^2}(s,t)\right| &\le C c_\rho^{-2}(s,t) J(s,t), \\ \left|\frac{\partial^3 J}{\partial s^3}(s,t)\right| &\le C c_\rho^{-3}(s,t) J(s,t). \end{aligned}$$

Proof. Fix $t \in (0, \infty)$ and write $J(s) = J(s,t)$, $K(s) = K(s,t)$ and $a = a_t$ for simplicity. The first estimate follows from Lemma 18. By (47) and (53),

$$J'' = -KJ = \frac{6}{c_\rho^2} J,$$

which is the second estimate. Finally, differentiate (53) with respect to s to obtain

$$J''' = -K'J - KJ'.$$

Since $c'_\rho \le |c'| = 1$, we have

$$|K'| = \left|\frac{\partial}{\partial s}\left(\frac{-6}{c_\rho^2(s,t)}\right)\right| = \frac{12}{c_\rho^3(s,t)} \left|c'_\rho\right| \le \frac{12}{c_\rho^3(s,t)}.$$

Combining the above with the previous inequalities, we obtain the third. □

Lemma 20. *For $c(s,t)$ given by* (11) *and $J(s,t) = |\frac{\partial c}{\partial t}(s,t)|$, there exists $C > 1$ such that for $(s,t) \in [-1,1] \times (-\infty, \frac{3}{2})$,*

$$\begin{aligned} \left|\frac{\partial J}{\partial t}(s,t)\right| &\le C c_\rho^{2}(s,t) J(s,t), \\ \left|\frac{\partial^2 J}{\partial t \partial s}(s,t)\right| &\le C c_\rho(s,t) J(s,t), \\ \left|\frac{\partial^3 J}{\partial t \partial s^2}(s,t)\right| &\le C J(s,t). \end{aligned}$$

Proof. Throughout this proof, we will use C to denote a generic constant that is independent of $(s,t) \in [-1,1] \times (-\infty, \frac{3}{2})$. For simplicity, we denote the t-derivative by a dot. Differentiate (47) with respect to t to obtain

$$\dot{K} = \frac{\partial}{\partial t}\left(-\frac{6}{c_\rho^2}\right) = \frac{12\dot{c}_\rho}{c_\rho^3}. \tag{54}$$

The definition of J and Lemma 18 implies

$$0 \leq \dot{c}_\rho \leq J \leq c_\rho^3,$$

and thus

$$0 \leq \dot{K} \leq 12. \tag{55}$$

Differentiate (53) with respect to t and note that $\dot{c}_\rho(0,t) = c_\rho^3(0,t)$ (cf. (12)) to obtain

$$\dot{J}'' = -K\dot{J} - \dot{K}J, \quad \dot{J}(0) = 3c_\rho^5(0,t), \quad \dot{J}'(0) = 0. \tag{56}$$

Combining (53) and (56), we obtain

$$(J'\dot{J} - J\dot{J}')' = \dot{K}J^2. \tag{57}$$

Define $\eta(s) := \frac{\dot{J}(s)}{J(s)}$. Since $s \mapsto J(s)$ is increasing in $[0,1]$ (cf. Lemma 18), $J(0) > 0$ and $J(s) = J(-s)$, the function η is well defined. By (55) and (57),

$$|(-\eta'(s)J^2)'| = \dot{K}J^2 \leq 12J^2.$$

By symmetry, $\eta'(0) = 0$. Since $s \mapsto J(s)$ is increasing in $[0,1]$, integration implies

$$|-\eta'(s)|J^2(s) \leq \int_0^s 12J^2 \leq 12sJ^2(s) \leq 12c_\rho(s,t)J^2(s) \quad \text{for all } s \in [0,1],$$

which simplifies to

$$|\eta'(s)| \leq 12c_\rho(s,t) \quad \text{for all } s \in [0,1].$$

By the initial conditions $J(0) = c_\rho^3(0,t)$ (cf. (53)) and $\dot{J}(0) = 3c_\rho^5(0,t)$ (cf. (56)), we obtain

$$\eta(0) = \frac{\dot{J}(0)}{J(0)} = 3c_\rho^2(0,t).$$

Since $s \mapsto c_\rho(s,t)$ is increasing in $[0,1]$, integration implies

$$|\eta(s)| \leq |\eta(0)| + \int_0^s |\eta'|\,ds \leq 3c_\rho^2(0,t) + 12sc_\rho(s,t) \leq 15c_\rho^2(s,t), \tag{58}$$

which in turn implies

$$|\dot{J}(s)| = |\eta(s)|J(s) \leq 15c_\rho^2(s,t)J(s).$$

This proves the first inequality of the lemma for $s \in [0,1]$. By symmetry, the same argument applies for $s \in [-1,0]$. By (47), (55), (56) and the first estimate,

$$|\dot{J}''(s)| = |-K\dot{J} - \dot{K}J| \leq CJ \quad \text{for all } s \in [-1,1].$$

Since $\dot{J}'(0) = 0$ and $\sigma \mapsto J(\sigma)$ is an increasing function in $[0,1]$, integration, Lemma 17 and symmetry imply

$$|\dot{J}'(s)| = \dot{J}'(0) + \int_0^{|s|} |\dot{J}''| \le C|s|J(s) \le C c_\rho(s,t) J(s) \quad \text{for all } s \in [0,1].$$

The above two inequalities complete the proof of the lemma. □

Lemma 21. *For $c(s,t)$ given by (11) and $J(s,t) = |\frac{\partial c}{\partial t}(s,t)|$, there exists $C > 1$ such that for $(s,t) \in [-1,1] \times (-\infty, \frac{3}{2})$,*

$$|\dot{c}_\rho| \le C c_\rho^3(s,t) \quad \textit{and} \quad |\ddot{c}_\rho| \le C c_\rho^5(s,t),$$

where $\dot{c}$ and $\ddot{c}$ indicate the first and second derivatives of c with respect to t, respectively.

Proof. Throughout this proof, we will use C to denote a generic constant that is independent of $(s,t) \in [-1,1] \times (-\infty, \frac{3}{2})$. For simplicity, we omit the subscript t and denote the t-derivative by a dot. We can write

$$\dot{c} = \dot{c}_\rho \frac{\partial}{\partial \rho} + \dot{c}_\phi \frac{\partial}{\partial \phi}$$

and

$$c' = c'_\rho \frac{\partial}{\partial \rho} + c'_\phi \frac{\partial}{\partial \phi},$$

where

$$|c'_\rho|, |c_\rho^3 c'_\phi| \le \sqrt{(c'_\rho)^2 + c_\rho^6 (c'_\phi)^2} = |c'| = 1 \tag{59}$$

and

$$|\dot{c}_\rho|, |c_\rho^3 \dot{c}_\phi| \le \sqrt{\dot{c}_\rho^2 + c_\rho^6 \dot{c}_\phi^2} = |\dot{c}| = J. \tag{60}$$

Lemma 18 and (60) imply

$$|\dot{c}_\rho| \le c_\rho^3(s,t) \quad \text{and} \quad |\dot{c}_\phi| \le 1 \quad \text{for all } s \in [-1,1]. \tag{61}$$

The first inequality above is the first inequality of the lemma. We now proceed with the proof of the second inequality of the lemma. Since $\langle \dot{c}, c' \rangle \equiv 0$ by (48), we obtain

$$\langle \ddot{c}, c' \rangle = \frac{\partial}{\partial t} \langle \dot{c}, c' \rangle - \langle \dot{c}, \dot{c}' \rangle = -\frac{\frac{\partial}{\partial s} \langle \dot{c}, \dot{c} \rangle}{2} = -JJ'.$$

Additionally,

$$\langle \ddot{c}, \dot{c} \rangle = \frac{\frac{\partial}{\partial t} \langle \dot{c}, \dot{c} \rangle}{2} = J\dot{J}.$$

Since

$$\left\langle c', \frac{\partial}{\partial \rho} \right\rangle = c'_\rho, \qquad \left\langle \frac{\dot{c}}{J}, \frac{\partial}{\partial \rho} \right\rangle = \frac{\dot{c}_\rho}{J},$$
$$\left\langle c', \frac{\partial}{\partial \phi} \right\rangle = c_\rho^6 c'_\phi, \quad \left\langle \frac{\dot{c}}{J}, \frac{\partial}{\partial \phi} \right\rangle = \frac{c_\rho^6 \dot{c}_\phi}{J},$$

we can express $\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial\phi}$ in terms of the orthonormal basis $\{c', \frac{\dot{c}}{J}\}$ (cf. (48)) by

$$\frac{\partial}{\partial\rho} = c'_\rho c' + \frac{\dot{c}_\rho}{J}\frac{\dot{c}}{J} \quad \text{and} \quad \frac{\partial}{\partial\phi} = c_\rho^6 c'_\phi c' + \frac{c_\rho^6 \dot{c}_\phi}{J}\frac{\dot{c}}{J}.$$

Combining the above identities, we obtain

$$\left\langle \ddot{c}, \frac{\partial}{\partial\rho}\right\rangle = \left\langle \ddot{c}, c'_\rho c' + \frac{\dot{c}_\rho}{J}\frac{\dot{c}}{J}\right\rangle = c'_\rho\langle\ddot{c}, c'\rangle + \frac{\dot{c}_\rho}{J^2}\langle\ddot{c},\dot{c}\rangle = -c'_\rho JJ' + \frac{\dot{c}_\rho \dot{J}}{J},$$
$$\left\langle \ddot{c}, \frac{\partial}{\partial\phi}\right\rangle = \left\langle \ddot{c}, c_\rho^6 c'_\phi c' + \frac{c_\rho^6\dot{c}_\phi}{J}\frac{\dot{c}}{J}\right\rangle = c_\rho^6 c'_\phi\langle\ddot{c}, c'\rangle + \frac{c_\rho^6\dot{c}_\phi}{J^2}\langle\ddot{c},\dot{c}\rangle = -c_\rho^6 c'_\phi JJ' + \frac{c_\rho^6\dot{c}_\phi \dot{J}}{J},$$
$$\left\langle \dot{c}', \frac{\partial}{\partial\rho}\right\rangle = \left\langle \dot{c}', c'_\rho c' + \frac{\dot{c}_\rho}{J}\frac{\dot{c}}{J}\right\rangle = c'_\rho\langle\dot{c}', c'\rangle + \frac{\dot{c}_\rho}{J^2}\langle\dot{c}',\dot{c}\rangle = \frac{\dot{c}_\rho}{J^2}\frac{\frac{\partial}{\partial s}\langle\dot{c},\dot{c}\rangle}{2} = \frac{\dot{c}_\rho J'}{J}.$$

By (2), we obtain

$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial\rho} = \dot{c}_\rho\nabla_{\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\rho} + \dot{c}_\phi\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\rho} = \dot{c}_\phi\Gamma^\phi_{\rho\phi}\frac{\partial}{\partial\phi} = \frac{3\dot{c}_\phi}{c_\rho}\frac{\partial}{\partial\phi},$$
$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial\phi} = \dot{c}_\rho\nabla_{\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\phi} + \dot{c}_\phi\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} = \dot{c}_\rho\Gamma^\phi_{\rho\phi}\frac{\partial}{\partial\phi} + \dot{c}_\phi\Gamma^\rho_{\phi\phi}\frac{\partial}{\partial\rho} = \frac{3\dot{c}_\rho}{c_\rho}\frac{\partial}{\partial\phi} - 3\dot{c}_\phi c_\rho^5\frac{\partial}{\partial\rho}.$$

Thus,

$$\ddot{c}_\rho = \frac{\partial}{\partial t}\left\langle\dot{c}, \frac{\partial}{\partial\rho}\right\rangle = \left\langle\ddot{c}, \frac{\partial}{\partial\rho}\right\rangle + \left\langle\dot{c}, \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial\rho}\right\rangle = -c'_\rho JJ' + \frac{\dot{c}_\rho\dot{J}}{J} + 3c_\rho^5\dot{c}_\phi^2. \tag{62}$$

Applying (59), (61), Lemma 18, Lemma 19 and Lemma 20 to the right-hand side of (62), we obtain the second inequality of the lemma. □

Lemma 22. *For $c(s,t)$ given by* (11) *and $J(s,t) = |\frac{\partial c}{\partial t}(s,t)|$, there exists $C > 1$ such that for $(s,t) \in [-1,1] \times (-\infty, \frac{3}{2})$,*

$$\left|\frac{\partial^2 J}{\partial t^2}(s,t)\right| \le Cc_\rho^4(s,t)J(s,t),$$
$$\left|\frac{\partial^3 J}{\partial s\partial t^2}(s,t)\right| \le Cc_\rho^3(s,t)J(s,t),$$
$$\left|\frac{\partial^4 J}{\partial s^2\partial t^2}(s,t)\right| \le Cc_\rho^2(s,t)J(s,t).$$

Proof. Throughout this proof, we will use C to denote a generic constant that is independent of $(s,t) \in [-1,1] \times (-\infty, \frac{3}{2})$. Differentiate (56) and note (12) to obtain

$$\ddot{J}'' = -K\ddot{J} - \ddot{K}J - 2\dot{K}\dot{J}, \quad \ddot{J}'(0) = 0, \quad \ddot{J}(0) = 15c_\rho^7(0,t). \tag{63}$$

Differentiate (54) to obtain

$$\ddot{K} = -\frac{36\dot{c}_\rho^2}{c_\rho^4} + \frac{12\ddot{c}_\rho}{c_\rho^3}. \tag{64}$$

Applying Lemma 21, we thus obtain

$$|\ddot{K}| \le C c_\rho^2(s,t). \tag{65}$$

By (65), (55) and Lemma 20, we obtain

$$|\ddot{K} J + 2\dot{K}\dot{J}| \le C c_\rho^2(s,t) J \quad \text{in } [-1,1]. \tag{66}$$

By (53), (63) and (66), we obtain

$$|(J'\ddot{J} - J\ddot{J}')'| = |(\ddot{K} J + 2\dot{K}\dot{J})J| \le C c_\rho^2(s,t) J^2. \tag{67}$$

For $s \in [0,1]$ let $\tau(s) = \frac{\ddot{J}(s)}{J(s)}$. Then (67) implies

$$|(-\tau'(s) J^2(s))'| \le C c_\rho^2(s,t) J^2(s).$$

By symmetry, $\tau'(0) = 0$. Since $\sigma \mapsto c_\rho(\sigma,t)$ and $\sigma \mapsto J(\sigma)$ are both increasing in $[0,1]$, integration and Lemma 17 implies

$$|\tau'(s)| J^2(s) \le C \int_0^s c_\rho^2(\sigma,t) J^2(\sigma)\, d\sigma \le C s c_\rho^2(s,t) J^2(s) \le C c_\rho^3(s,t) J^2(s),$$

which simplifies to

$$|\tau'(\sigma)| \le C c_\rho^3(\sigma,t).$$

Using (53) and (63), we thus obtain

$$|\tau(0)| = \left| \frac{\ddot{J}(0)}{J(0)} \right| = 15 c_\rho^4(0,t).$$

Thus, integration and Lemma 17 imply

$$|\tau(s)| \le |\tau(0)| + \int_0^s |\tau'(\sigma)|\, d\sigma \le C c_\rho^4(s,t),$$

which in turn implies

$$|\ddot{J}(s)| = |\tau(s)| J(s) \le C c_\rho^4(s,t) J(s).$$

This proves the first estimate of the lemma for $s \in [0,1]$. By symmetry, the same argument applies for $s \in [-1,0]$. By (63), (66) and the first estimate of the lemma,

$$|\ddot{J}''(s)| = |-K\ddot{J} - \ddot{K} J - 2\dot{K}\dot{J}| \le C c_\rho^2(s,t) J \quad \text{for all } s \in [-1,1].$$

Since $\dot{J}'(0) = 0$, integration and Lemma 18 imply

$$|\ddot{J}'(s)| \le C c_\rho^3(s,t) J(s) \quad \text{for all } s \in [-1,1].$$

The above two inequalities are the third and second estimates of the lemma. □

Lemma 23. *For $c(s,t)$ given by* (11)*, there exists $C > 1$ such that*

$$|\check{c}_\rho| \le C c_\rho^7(s,t),$$

where $\check{c}$ indicates the third derivative of c with respect to t.

Proof. Continuing the computation in the proof of Lemma 21, we obtain

$$\ddot{c}_\phi = \frac{\partial}{\partial t}\Big\langle \dot{c}, c_\rho^{-6}\frac{\partial}{\partial\phi}\Big\rangle = c_\rho^{-6}\Big\langle \ddot{c}, \frac{\partial}{\partial\phi}\Big\rangle + \Big\langle \dot{c}, \nabla_{\frac{\partial}{\partial t}} c_\rho^{-6}\frac{\partial}{\partial\phi}\Big\rangle \tag{68}$$
$$= -c'_\phi JJ' + \frac{\dot{c}_\phi \dot{J}}{J} - \frac{6}{c_\rho}\dot{c}_\rho \dot{c}_\phi$$

and

$$\dot{c}'_\rho = \frac{\partial}{\partial t}\Big\langle c', \frac{\partial}{\partial\rho}\Big\rangle = \Big\langle \dot{c}', \frac{\partial}{\partial\rho}\Big\rangle + \Big\langle c', \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial\rho}\Big\rangle = \frac{\dot{c}_\rho J'}{J} + 3c_\rho^5 \dot{c}_\phi c'_\phi. \tag{69}$$

Differentiating (62) with respect to t and applying (59), (61), (68), (69), Lemma 18, Lemma 19, Lemma 20 and Lemma 22, we obtain the third estimate. □

Lemma 24. *For $c(s,t)$ given by* (11) *and $J(s,t) = |\frac{\partial c}{\partial t}(s,t)|$, there exists $C > 1$ such that for $(s,t) \in [-1,1] \times (-\infty, \frac{3}{2})$,*

$$\Big|\frac{\partial^3 J}{\partial t^3}(s,t)\Big| \le C c_\rho^6(s,t) J(s,t),$$
$$\Big|\frac{\partial^4 J}{\partial s \partial t^3}(s,t)\Big| \le C c_\rho^5(s,t) J(s,t),$$
$$\Big|\frac{\partial^5 J}{\partial s^2 \partial t^3}(s,t)\Big| \le C c_\rho^4(s,t) J(s,t).$$

Proof. Throughout this proof, we will use C to denote a generic constant that is independent of $(s,t) \in [-1,1] \times (-\infty, \frac{3}{2})$. Differentiate (63) to obtain (with the third derivative of J and K with respect to t denoted by $\check{J}$ and $\check{K}$, respectively)

$$\check{J}'' = -K\check{J} - \check{K}J - 3\ddot{K}\dot{J} - 3\dot{K}\ddot{J}, \quad \check{J}'(0) = 0, \quad \check{J}(0) = 105a^9. \tag{70}$$

Differentiate (64) to obtain

$$\check{K} = \frac{144\dot{c}_\rho^3}{c_\rho^5} - \frac{108\dot{c}_\rho \ddot{c}_\rho}{c_\rho^4} + \frac{12\check{c}_\rho}{c_\rho^3}. \tag{71}$$

Applying Lemma 21 and Lemma 23, we thus obtain

$$|\check{K}| \le C c_\rho^4(s,t) \quad \text{in } [-1,1]. \tag{72}$$

Thus, (72), (65), Lemma 20, (55) and Lemma 22 imply

$$|\check{K}J + 3\ddot{K}\dot{J} + 3\dot{K}\ddot{J}| \le C c_\rho^4(s,t) J \quad \text{in } [-1,1]. \tag{73}$$

Next, combining (53) and (70), we obtain

$$|(J'\check{J} - \check{J}'J)'| = |(\check{K}J + 3\ddot{K}\dot{J} + 3\dot{K}\ddot{J})J| \le C c_\rho^4(s,t) J^2 \quad \text{in } [-1,1]. \tag{74}$$

By comparing (73) to (66) and (74) to (67), we observe that the lemma follows from the proof of Lemma 22. □

Lemma 25. *For $c(s,t)$ given by* (11) *and $J(s,t) = |\frac{\partial c}{\partial t}(s,t)|$ we have*

$$e^{-\frac{3}{2}} c_\rho^3(s,t) \le J(s,t) \le c_\rho^3(s,t)$$

for $(s,t) \in [-1,1] \times (-\infty, \frac{3}{2})$.

Proof. By Lemma 18, it is suffices to show $e^{-\frac{3}{2}} \le J(1,t)$. Fix $t \in (0,\infty)$ and omit the subscript t for simplicity. With this notation, we rewrite (50) as

$$J'' + KJ = 0, \quad J(0) = c_\rho^3(0,t), \quad J'(0) = 0. \tag{75}$$

The function

$$j_1(s) = (s + c_\rho(0,t))^3$$

is the solution to the differential equation

$$j_1'' + k_1 j_1 = 0, \quad k_1 = -\frac{6}{(s + c_\rho(0,t))^2}$$

with boundary conditions

$$j_1(0) = c_\rho^3(0,t) \quad \text{and} \quad j_1'(0) = 3c_\rho^2(0,t).$$

For $s \in [0,1]$, Lemma 17 and (47) implies

$$-k_1 \le -K.$$

Hence

$$(J(s) j_1'(s) - J'(s) j_1(s))' \le 0. \tag{76}$$

Since $J'(0) = 0$ (cf. (75)), we have

$$J(0) j_1'(0) - J'(0) j_1(0) = J(0) j_1'(0),$$

which in turn implies by (76) that

$$J(s) j_1'(s) - J'(s) j_1(s) \le J(0) j_1'(0). \tag{77}$$

Since $J'(s) \ge 0$ for $s \in [0,1]$, we have that $J(0) \le J(s)$ and

$$\begin{aligned} \frac{\partial}{\partial s}\left(\log \frac{j_1(s)}{J(s)}\right) &= \frac{j_1'(s) J(s) - J'(s) j_1(s)}{j_1(s) J(s)} \\ &\overset{(77)}{\le} \frac{j_1'(0) J(0)}{j_1(s) J(s)} \\ &\le \frac{j_1'(0)}{j_1(s)} \\ &= \frac{3c_\rho^2(0,t)}{(s + c_\rho(0,t))^3} \quad \text{for all } s \in [0,1]. \end{aligned} \tag{78}$$

Integrating this inequality in the interval $[0,1]$, we obtain

$$\frac{j_1(1)}{J(1)} \cdot \frac{J(0)}{j_1(0)} \le \exp\left(-\frac{3c_\rho^2(0,t)}{2}\left(\frac{1}{(1 + c_\rho(0,t))^2} - \frac{1}{c_\rho^2(0,t)}\right)\right) \le e^{\frac{3}{2}} \quad \text{for all } s \in [0,1].$$

Since $j_1(0) = c_\rho^3(0,t) = J(0)$, we obtain

$$e^{-\frac{3}{2}} \leq e^{-\frac{3}{2}} j_1(1) \leq J(1),$$

which is the required lower bound for $J(1)$. □

Lemma 26. *For* $c(s,t)$ *given by* (11) *and* $J(s,t) = |\frac{\partial c}{\partial t}(s,t)|$, *there exists* $C > 1$ *such that for* $(s,t) \in [-1,1] \times (-\infty,1]$,

$$C^{-1} c_\rho^3(s,t) \leq J(s,t) \leq c_\rho^3(s,t).$$

Proof. Applying the first estimate of Lemma 20 with $s = 1$, we obtain

$$\frac{\dot{J}(1,t)}{J(1,t)} \leq C c_\rho^2(1,t) \leq C c_\rho^2(1,1) =: C_0 \quad \text{for all } t \in (-\infty,1].$$

Integration implies

$$\log \frac{J(1,1)}{J(1,t)} \leq C_0(1-t) \leq C_0 \quad \text{for all } t \in (-\infty,1].$$

Thus, by Lemma 18,

$$e^{-C_0} J(1,1) c_\rho^3(s,t) \leq J(1,t) c_\rho^3(s,t) \leq J(s,t) \leq c_\rho^3(s,t)$$

for $s \in [-1,1]$ and $t \in (-\infty,1]$. □

5. The metric estimates

In this section, we use the family of symmetric geodesics $c(s,t)$ given in the previous section to define a new coordinate system for $\mathbf{H}$. We then give estimates of the metric $g_{\mathbf{H}}$ and its Christoffel symbols represented in terms these new coordinates.

For a fixed $t_* < 0$, apply a linear change of variables

$$(s,t) \mapsto (\varrho, \varphi) = (s, t - t_*). \tag{79}$$

Our goal is to compare the geometry of our space $\mathbf{H}$ near $(0,0)$ to that of the space $\overline{\mathbf{H}}_2$ near P_0. The metric and the Christoffel symbols of $\overline{\mathbf{H}}_2 \backslash \{P_0\}$ expressed in the coordinates (ρ, ϕ) are given by

$$\begin{pmatrix} 1 & 0 \\ 0 & \rho^6 \end{pmatrix} \quad \text{and} \quad \begin{matrix} \Gamma^\rho_{\rho\rho} = 0, & \Gamma^\phi_{\phi\phi} = 0, \\ \Gamma^\rho_{\rho\phi} = 0, & \Gamma^\phi_{\rho\phi} = \dfrac{3}{\rho}, \\ \Gamma^\rho_{\phi\phi} = -3\rho^5, & \Gamma^\phi_{\rho\rho} = 0. \end{matrix}$$

We want to compare the above metric expression and the Christoffel symbols to that of $g_{\mathbf{H}}$ in terms of our new coordinates given in (79). By construction (cf. (14), (43) and (48)), the metric $g_{\mathbf{H}}$ with respect to coordinates (ϱ, φ) of $\overline{\mathbf{H}}_2 \backslash \{P_0\}$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{J}^2(\varrho, \varphi) \end{pmatrix},$$

where we set

$$\mathcal{J}(\varrho,\varphi) := \left|\frac{\partial c}{\partial t}(\varrho,\varphi+t_*)\right| = J(\varrho,\varphi+t_*)$$

and

$$\Gamma^{\varrho}_{\varrho\varrho}=0, \qquad \Gamma^{\varphi}_{\varphi\varphi}=\frac{\partial}{\partial\varphi}\log\mathcal{J},$$
$$\Gamma^{\varrho}_{\varrho\varphi}=0, \qquad \Gamma^{\varphi}_{\varrho\varphi}=\frac{\partial}{\partial\varrho}\log\mathcal{J},$$
$$\Gamma^{\varrho}_{\varphi\varphi}=-\frac{1}{2}\frac{\partial}{\partial\varrho}\mathcal{J}^2=-\frac{\partial\mathcal{J}}{\partial\varrho}\mathcal{J}, \quad \Gamma^{\varphi}_{\varrho\varrho}=0. \tag{80}$$

Since we are interested in the geometry of $\overline{\mathbf{H}}$ near P_0, we will assume

$$(\varrho,\varphi)\in[-1,1]\times(-\infty,1-t_*].$$

We thus can apply Lemma 19 through Lemma 24. For simplicity, introduce the function

$$\wp=\wp(\varrho,\varphi)$$

given by

$$\wp=c_\rho(\varrho,\varphi+t_*). \tag{81}$$

By Lemma 17 and Lemma 18, there exists a constant $C\geq 1$ such that

$$C^{-1}\varrho^6\leq C^{-1}\wp^6\leq\mathcal{J}^2(\varrho,\varphi)\leq\wp^6. \tag{82}$$

Furthermore, Lemma 19, Lemma 20, Lemma 22, Lemma 24 and the fact that $(s,t)\mapsto(\varrho,\varphi)$ is a linear change of variables (and thus $\frac{\partial}{\partial s}=\frac{\partial}{\partial\varrho}$, $\frac{\partial}{\partial t}=\frac{\partial}{\partial\varphi}$) yield the *derivative estimates of* $\mathcal{J}$; i.e. there exists $C\geq 1$ such that

$$\begin{aligned}
&\left|\frac{\partial\mathcal{J}}{\partial\varrho}(\varrho,\varphi)\right|\leq C\wp^2, &&\left|\frac{\partial^2\mathcal{J}}{\partial\varrho^2}(\varrho,\varphi)\right|\leq C\wp, &&\left|\frac{\partial^3\mathcal{J}}{\partial\varrho^3}(\varrho,\varphi)\right|\leq C,\\
&\left|\frac{\partial\mathcal{J}}{\partial\varphi}(\varrho,\varphi)\right|\leq C\wp^5, &&\left|\frac{\partial^2\mathcal{J}}{\partial\varphi\partial\varrho}(\varrho,\varphi)\right|\leq C\wp^4, &&\left|\frac{\partial^3\mathcal{J}}{\partial\varphi\partial\varrho^2}(\varrho,\varphi)\right|\leq C\wp^3,\\
&\left|\frac{\partial^2\mathcal{J}}{\partial\varphi^2}(\varrho,\varphi)\right|\leq C\wp^7, &&\left|\frac{\partial^3\mathcal{J}}{\partial\varphi^2\partial\varrho}(\varrho,\varphi)\right|\leq C\wp^6, &&\left|\frac{\partial^4\mathcal{J}}{\partial\varphi^2\partial\varrho^2}(\varrho,\varphi)\right|\leq C\wp^5,\\
&\left|\frac{\partial^3\mathcal{J}}{\partial\varphi^3}(\varrho,\varphi)\right|\leq C\wp^9, &&\left|\frac{\partial^4\mathcal{J}}{\partial\varphi^3\partial\varrho}(\varrho,\varphi)\right|\leq C\wp^8, &&\left|\frac{\partial^5\mathcal{J}}{\partial\varphi^3\partial\varrho^2}(\varrho,\varphi)\right|\leq C\wp^7.
\end{aligned}\tag{83}$$

We can apply the above estimate to obtain the following *Christoffel symbols estimates*: there exists $C\geq 1$ such that

$$\begin{aligned}
&\left|\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi)\leq C\wp^5, \qquad \left|\Gamma^{\varphi}_{\varrho\varphi}\right|(\varrho,\varphi)\leq\frac{C}{\wp}, \qquad \left|\Gamma^{\varphi}_{\varphi\varphi}\right|(\varrho,\varphi)\leq C\wp^2,\\
&\left|\frac{\partial}{\partial\varrho}\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi)\leq C\wp^4, \qquad \left|\frac{\partial^2}{\partial\varrho^2}\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi)\leq C\wp^3,\\
&\left|\frac{\partial}{\partial\varphi}\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi)\leq C\wp^7, \qquad \left|\frac{\partial^2}{\partial\varphi^2}\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi)\leq C\wp^9,\\
&\left|\frac{\partial^2}{\partial\varphi\partial\varrho}\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi)\leq C\wp^6.
\end{aligned}\tag{84}$$

6. Essential regularity of $\overline{\mathbf{H}}$

The goal of this section is to prove the essential regularity of $\overline{\mathbf{H}}$. This is the analogue of Theorem 15. Fix $t_* < 0$ and let (ϱ, φ) be coordinates of $\mathbf{H}$ as in (79). For a harmonic map $v : (B_1(0), g) \to (\mathbf{H}, d_{\mathbf{H}})$, write

$$v = (v_\varrho, v_\varphi)$$

in our new coordinates (ϱ, φ). For simplicity, set $\wp_v$ to be the composition of $\wp$ defined in (81) and the map v; i.e.

$$\wp_v := \wp(v_\varrho, v_\varphi) = c_\rho(v_\varrho, v_\varphi + t_*). \tag{85}$$

Fix $R \in [\frac{3}{4}, 1)$. The map v is locally Lipschitz continuous (cf. [15, Theorem 2.4.6]); in particular,

$$|\nabla v| = \big(|\nabla v_\varrho|^2 + |\mathcal{J}(v_\varrho, v_\varphi)|^2 |\nabla v_\varphi|^2\big)^{\frac{1}{2}} \le L \quad \text{in } B_R(0)$$

for some constant L that depends on the dimension n of the domain, R and the total energy of v. Thus, (82) implies

$$|\nabla v_\varrho| \le L \quad \text{and} \quad \wp_v^3 |\nabla v_\varphi| \le L \quad \text{in } B_R(0). \tag{86}$$

The harmonic map equations are (cf. (80))

$$\triangle v_\varrho = -\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2 = \frac{\partial \mathcal{J}}{\partial \varrho} \mathcal{J} |\nabla v_\varphi|^2 \tag{87}$$

and

$$\begin{aligned} \triangle v_\varphi &= -(2\Gamma^\varphi_{\varrho\varphi}(v_\varrho, v_\varphi)\nabla v_\varphi \cdot \nabla v_\varrho + \Gamma^\varphi_{\varphi\varphi}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2) \\ &= -\frac{(2\frac{\partial \mathcal{J}}{\partial \varrho}(v_\varrho, v_\varphi)\nabla v_\varphi \cdot \nabla v_\varrho + \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2)}{\mathcal{J}(v_\varrho, v_\varphi)} \\ &= -\frac{(2\nabla \mathcal{J}(v_\varrho, v_\varphi) \cdot \nabla v_\varphi - \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2)}{\mathcal{J}(v_\varrho, v_\varphi)}. \end{aligned} \tag{88}$$

Thus, the Christoffel symbols estimates (84) and Lipschitz estimates (86) imply

$$\begin{aligned} |\triangle v_\varrho| &\le C\wp_v^5 |\nabla v_\varphi|^2 \le \frac{CL^2}{\wp_v}, \\ |\triangle v_\varphi| &\le C(\wp_v^{-1}|\nabla v_\varrho||\nabla v_\varphi| + \wp_v^2 |\nabla v_\varphi|^2) \le \frac{CL^2}{\wp_v^4} \end{aligned} \tag{89}$$

in $B_R(0)$. In analogy with (31) we define

$$v_\Upsilon := v_\varrho + \frac{v_\varphi^2}{2}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi) \quad \text{and} \quad v_\Phi := \mathcal{J}(v_\varrho, v_\varphi)v_\varphi - \frac{1}{2}\frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)v_\varphi^2. \tag{90}$$

For $\varphi_0 > 0$, define the set

$$\mathbf{H}[\varphi_0] := \{(\varrho, \varphi) \in \mathbf{H} : |\varphi| \le \varphi_0\}.$$

Note since the level sets $\varphi = \varphi_0$ and $\varphi = -\varphi_0$ are images of the geodesic lines $\varrho \mapsto c(\varrho, \varphi_0 + t_0)$ and $\varrho \mapsto c(\varrho, -\varphi_0 + t_0)$, the set $\mathbf{H}[\varphi_0]$ is geodesically convex. We also define

$$a[\varphi_0] := \max_{\{\varphi : |\varphi| \leq \varphi_0\}} c_\rho(0, \varphi + t_*) = c_\rho(0, \varphi_0 + t_*).$$

In particular, we have

$$\begin{aligned} \wp &= c_\rho(\varrho, \varphi + t_*) \\ &= d_{\overline{\mathbf{H}}}(c(\varrho, \varphi + t_*), P_0) \\ &\leq d_{\overline{\mathbf{H}}}(c(\varrho, \varphi + t_*), c(0, \varphi + t_*)) + d_{\overline{\mathbf{H}}}(c(0, \varphi + t_*), P_0) \\ &\leq \varrho + a[\varphi_0] \quad \text{in } \mathbf{H}[\varphi_0]. \end{aligned} \tag{91}$$

We are now in a position to prove the following analogue of Lemma 13. We will denote a geodesic ball of radius R centered at P_0 in $\overline{\mathbf{H}}$ by $\mathbf{B}_R(P_0)$.

Lemma 27. *Let $R \in [\frac{1}{2}, 1)$, $E_0 > 0$, $A_0 > 0$ and a normalized metric g on $B_R(0)$ be given. There exist $C_0 \geq 1$ depending only on E_0, A_0 and g with the following property: For $\phi_0 > 0$, $\sigma \in (0, 1]$, $\vartheta \in (0, 1]$, if*

$$v = (v_\rho, v_\phi) : (B_{\vartheta R}(0), g_\sigma) \to \overline{\mathbf{H}}\left[\frac{\phi_0}{\vartheta^2}\right] \cap \mathbf{B}_{A_0\vartheta}(P_0)$$

is a harmonic map with

$$a\left[\frac{\varphi_0}{\vartheta^2}\right] \leq \frac{\vartheta}{2} \tag{92}$$

and

$$E^v \leq \vartheta^n E_0,$$

then

$$|\mathcal{J}(v_\varrho, v_\varphi) \nabla v_\varphi|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} \leq C_0 \phi_0$$

and

$$|\triangle v_\Upsilon|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} \leq C_0 \vartheta^{-1} \phi_0^2.$$

Proof. Throughout the proof, C_0 will denote a generic constant dependent only on E_0, A_0 and g. First (91) and (92) imply that

$$\wp_v \leq v_\varrho + a\left[\frac{\phi_0}{\vartheta^2}\right] \leq C_0 \vartheta. \tag{93}$$

Combining this with (82), (83) and (90), we obtain

$$|v_\Phi|_{L^\infty(B_R(0))} \leq C \vartheta \phi_0. \tag{94}$$

Since the harmonic map equation (88) implies

$$2\nabla \mathcal{J}(v_\varrho, v_\varphi) \cdot \nabla v_\varphi + \mathcal{J}(v_\varrho, v_\varphi) \triangle v_\varphi - \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi) |\nabla v_\varphi|^2 = 0,$$

we have

$$\begin{aligned}(95)\qquad \triangle v_\Phi &= \triangle\left(\mathcal{J}(v_\varrho, v_\varphi)v_\varphi - \frac{1}{2}\frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)v_\varphi^2\right)\\ &= \triangle \mathcal{J}(v_\varrho, v_\varphi)v_\varphi + 2\nabla \mathcal{J}(v_\varrho, v_\varphi)\cdot \nabla v_\varphi + \mathcal{J}(v_\varrho, v_\varphi)\triangle v_\varphi\\ &\quad - \frac{1}{2}\triangle \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)v_\varphi^2 - 2\nabla \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)\cdot \nabla v_\varphi v_\varphi\\ &\quad - \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)\triangle v_\varphi v_\varphi - \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2\\ &= \triangle \mathcal{J}(v_\varrho, v_\varphi)v_\varphi - \frac{1}{2}\triangle \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)v_\varphi^2\\ &\quad - 2\nabla \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)\cdot \nabla v_\varphi v_\varphi - \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)\triangle v_\varphi v_\varphi.\end{aligned}$$

We now estimate the terms on the right-hand side of (95). First,

$$(96)\quad \triangle \mathcal{J}(v_\varrho, v_\varphi) = \frac{\partial \mathcal{J}}{\partial \varrho}\triangle v_\varrho + \frac{\partial^2 \mathcal{J}}{\partial \varrho^2}|\nabla v_\varrho|^2 + \frac{\partial \mathcal{J}}{\partial \varphi}\triangle v_\varphi + \frac{\partial^2 \mathcal{J}}{\partial \varphi^2}|\nabla v_\varphi|^2 + 2\frac{\partial^2 \mathcal{J}}{\partial \varphi \partial \varrho}\nabla v_\varrho \cdot \nabla v_\varphi.$$

By the metric derivative estimates (83), the Lipschitz estimate (86) and the bounds of the Laplacians (89), the five terms on the right-hand side are bounded in $B_R(0)$ by $CL^2\wp_v$. Thus, by (93) we conclude

$$(97)\qquad \triangle \mathcal{J}(v_\varrho, v_\varphi) \le C\wp_v L^2 \le C_0\vartheta \quad \text{in } B_{\vartheta R}(0).$$

Similarly,

$$\begin{aligned}(98)\qquad \left|\triangle \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)\right| &= \left|\frac{\partial^2 \mathcal{J}}{\partial \varphi \partial \varrho}\triangle v_\varrho + \frac{\partial^3 \mathcal{J}}{\partial \varphi \partial \varrho^2}|\nabla v_\varrho|^2 + \frac{\partial^2 \mathcal{J}}{\partial \varphi^2}\triangle v_\varphi\right.\\ &\quad \left. + \frac{\partial^3 \mathcal{J}}{\partial \varphi^3}|\nabla v_\varphi|^2 + 2\frac{\partial^3 \mathcal{J}}{\partial \varphi^2 \partial \varrho}\nabla v_\varrho \cdot \nabla v_\varphi\right|\\ &\le C\wp_v^3 L^2 \le C_0\vartheta^3 \quad \text{in } B_{\vartheta R}(0),\end{aligned}$$

$$\begin{aligned}(99)\qquad \left|\nabla \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)\cdot \nabla v_\varphi\right| &= \left|\frac{\partial^2 \mathcal{J}}{\partial \varrho \partial \varphi}(v_\varrho, v_\varphi)\nabla v_\varrho \cdot \nabla v_\varphi + \frac{\partial^2 \mathcal{J}}{\partial \varphi^2}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2\right|\\ &\le C\wp_v L^2 \le C_0\vartheta \quad \text{in } B_{\vartheta R}(0),\end{aligned}$$

$$(100)\qquad \left|\frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)\triangle v_\varphi\right| \le C\wp_v L^2 \le C_0\vartheta \quad \text{in } B_{R\vartheta}(0).$$

By (95), (97), (98), (99) and (100),

$$(101)\qquad |\triangle v_\Phi| \le C_0\varphi_0\vartheta^{-1} \quad \text{in } B_{\vartheta R}(0).$$

Thus, by (94), (101) and elliptic regularity, for any $\alpha \in (0, 1)$,

$$(102)\quad \vartheta|v_\Phi|_{C^{1,\alpha}(B_{\frac{15\vartheta R}{16}}(0))} \le C_0(\vartheta^2|\triangle v_\Phi|_{L^\infty(B_{\vartheta R}(0))} + |v_\Phi|_{L^\infty(B_{\vartheta R}(0))}) \le C_0\phi_0.$$

Since

$$\nabla v_\Phi = \mathcal{J}(v_\varrho, v_\varphi)\nabla v_\varphi + \frac{\partial \mathcal{J}}{\partial \varrho}(v_\varrho, v_\varphi)\nabla v_\varrho v_\varphi + \frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)\nabla v_\varphi v_\varphi,$$

we have

$$|\mathcal{J}(v_\varrho, v_\varphi)\nabla v_\phi| \le \left|\frac{\partial \mathcal{J}}{\partial \varrho}(v_\varrho, v_\varphi)\nabla v_\varrho v_\varphi\right| + \left|\frac{\partial \mathcal{J}}{\partial \varphi}(v_\varrho, v_\varphi)\nabla v_\varphi v_\varphi\right| + |\nabla v_\Phi|.$$

By (83), the first two terms are bounded by $C_0\phi_0\vartheta$. By (102), the third term is also bounded by $C_0\phi_0\vartheta$. Thus, we obtain first estimate of the lemma. We also have

$$\wp_v^3|\nabla v_\varphi| \le C_0\varphi_0\vartheta \quad \text{in } B_{\frac{15\vartheta R}{16}}(0). \tag{103}$$

Next, we compute

$$\begin{aligned}
\triangle v_\Upsilon = {} & \triangle v_\varrho + \Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2 + v_\varphi \Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)\triangle v_\varphi \\
& + \frac{v_\varphi^2}{2}\frac{\partial}{\partial\varphi}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)\triangle v_\varphi + \frac{v_\varphi^2}{2}\frac{\partial^2}{\partial\varphi^2}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2 \\
& + \frac{v_\varphi^2}{2}\frac{\partial}{\partial\varrho}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)\triangle v_\varrho + \frac{v_\varphi^2}{2}\frac{\partial^2}{\partial\varrho^2}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)|\nabla v_\varrho|^2 \\
& + v_\varphi^2\frac{\partial^2}{\partial\varrho\partial\varphi}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)\nabla v_\varrho\cdot\nabla v_\varphi + 2v_\varphi\frac{\partial}{\partial\varrho}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)\nabla v_\varrho\cdot\nabla v_\varphi \\
& + 2v_\varphi\frac{\partial}{\partial\varphi}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2.
\end{aligned} \tag{104}$$

The harmonic map equation (87) implies that the first two terms of (104) cancel. Combining with (84), (89) and (103),

$$\begin{aligned}
|v_\varphi\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)\triangle v_\varphi| &\le C\frac{\varphi_0}{\vartheta^2}\wp_v^5(\wp_v^{-1}|\nabla v_\varrho||\nabla v_\varphi| + \wp_v^2|\nabla v_\varphi|^2) \le C_0\vartheta^{-1}\varphi_0^2, \\
\left|\frac{v_\varphi^2}{2}\frac{\partial}{\partial\varphi}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)\triangle v_\varphi\right| &\le C\frac{\varphi_0^2}{\vartheta^4}\wp_v^3L^2 \le C_0\vartheta^{-1}\varphi_0^2, \\
\left|\frac{v_\varphi^2}{2}\frac{\partial^2}{\partial\varphi^2}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2\right| &\le C\frac{\varphi_0^2}{\vartheta^4}\wp_v^3L^2 \le C_0\vartheta^{-1}\varphi_0^2, \\
\left|\frac{v_\varphi^2}{2}\frac{\partial}{\partial\varrho}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)\triangle v_\varrho\right| &\le C\frac{\varphi_0^2}{\vartheta^4}\wp_v^3L^2 \le C_0\vartheta^{-1}\varphi_0^2, \\
\left|\frac{v_\varphi^2}{2}\frac{\partial^2}{\partial\varrho^2}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)|\nabla v_\varrho|^2\right| &\le C\frac{\varphi_0^2}{\vartheta^4}\wp_v^3L^2 \le C_0\vartheta^{-1}\varphi_0^2, \\
\left|\frac{v_\varphi^2}{2}\frac{\partial^2}{\partial\varrho\partial\varphi}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)\nabla v_\varrho\cdot\nabla v_\varphi\right| &\le C\frac{\varphi_0^2}{\vartheta^4}\wp_v^3L^2 \le C_0\vartheta^{-1}\varphi_0^2, \\
\left|v_\varphi\frac{\partial}{\partial\varrho}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)\nabla v_\varrho\cdot\nabla v_\varphi\right| &\le C\frac{\varphi_0}{\vartheta^2}\wp_v^4|\nabla v_\varrho||\nabla v_\varphi| \le C_0\vartheta^{-1}\varphi_0^2, \\
\left|v_\varphi\frac{\partial}{\partial\varphi}\Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi)|\nabla v_\varphi|^2\right| &\le C\frac{\varphi_0}{\vartheta^2}\wp_v^7|\nabla v_\varphi|^2 \le C_0\vartheta^{-1}\varphi_0^2.
\end{aligned}$$

In summary, we have shown that there exists a constant $C_0 > 0$ depending only on R and the total energy of v such that

$$|\triangle v_\Upsilon| \le C_0\vartheta^{-1}\varphi_0^2 \quad \text{in } B_{\frac{15\vartheta R}{16}}(0), \tag{105}$$

which is the second estimate of the lemma. □

We are now ready to prove the main theorem of the section.

Theorem 28. *Let $R \in [\frac{1}{2}, 1)$, $E_0 > 0$, $A_0 > 0$ and a normalized metric g on $B_R(0)$ be given. There exist $C \geq 1$ and $\alpha > 0$ depending only on E_0, A_0 and g with the following property: For $\varphi_0 > 0$, $s \in (0,1]$, $\vartheta \in (0,1]$, if*

$$v : (B_{\vartheta R}(0), g_s) \to \overline{\mathbf{H}}\left[\frac{\varphi_0}{\vartheta^2}\right] \cap \mathbf{B}_{A_0\vartheta}(P_0)$$

is a harmonic map with

$$a\left[\frac{\varphi_0}{\vartheta^2}\right] \leq \frac{\vartheta}{2} \tag{106}$$

and

$$E^v \leq \vartheta^n E_0,$$

then

$$\sup_{B_{r\vartheta}(0)} d(v, \hat{l}) \leq C r^{1+\alpha} \sup_{B_{\vartheta R}(0)} d(v, L) + C r \vartheta \varphi_0^2 \quad \textit{for all } r \in \left(0, \frac{R}{2}\right], \tag{107}$$

where $\hat{l} = (\hat{l}_\rho, \hat{l}_\varphi) : B_1(0) \to \overline{\mathbf{H}}$ is the almost affine map given by

$$\hat{l}_\rho(x) = v_\rho(0) + \nabla v_\rho(0) \cdot x, \quad \hat{l}_\varphi(x) = v_\varphi(x)$$

and $L : B_1(0) \to \overline{\mathbf{H}}$ is any almost affine map.

Remark 29. A subset $E \subset \overline{\mathbf{H}}$ will be called *essentially regular* if any harmonic map $v : \Omega \to E$ from a Riemannian domain satisfies (107) above.

Proof. The proof is analogous to that of Theorem 15. Throughout the proof, let C be a generic constant that depends on E_0, A_0 and g. Since

$$v_\Upsilon - \hat{l}_\rho = v_\rho - \hat{l}_\rho - \frac{v_\varphi^2}{2} \Gamma^\varrho_{\varphi\varphi}(v_\varrho, v_\varphi),$$

equations (84) and (93) imply

$$\big| |v_\Upsilon - \hat{l}_\rho|_{L^\infty(B_{R\vartheta}(0))} - |v_\rho - \hat{l}_\rho|_{L^\infty(B_{R\vartheta}(0))} \big| \leq C \vartheta^5 \left(\frac{\varphi_0}{\vartheta^2}\right)^2 = C \vartheta \varphi_0^2.$$

Thus, elliptic regularity and the second estimate of Lemma 27 imply that

$$\begin{aligned}\vartheta^{1+\alpha}[\nabla(v_\Upsilon - \hat{l}_\rho)]_{C^\alpha(B_{\frac{7\vartheta R}{8}}(0))} &\leq C(\vartheta^2 |\triangle(v_\Upsilon - \hat{l}_\rho)|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} + |v_\Upsilon - \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))}) \\ &\leq C \vartheta \varphi_0^2 + C |v_\rho - \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))}.\end{aligned}$$

Since $\nabla v_\rho(0) = \nabla \hat{l}_\rho(0)$, we have

$$\begin{aligned}\sup_{B_{r\vartheta}(0)} |\nabla(v_\Upsilon - \hat{l}_\rho)| &\leq C(r\vartheta)^\alpha (\vartheta^{-\alpha} \varphi_0^2 + \vartheta^{-1-\alpha} |v_\rho - \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))}) \\ &\leq C r^\alpha (\varphi_0^2 + \vartheta^{-1} |v_\rho - \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))}).\end{aligned} \tag{108}$$

Furthermore, since

$$\nabla(v_\Upsilon - \hat{l}_\rho) = \nabla(v_\rho - \hat{l}_\rho) + \Gamma^{\varrho}_{\varphi\varphi}(v_\varrho, v_\varphi) v_\varphi \nabla v_\varphi - \frac{v_\varphi^2}{2}\frac{\partial}{\partial\varrho}\Gamma^{\varrho}_{\varphi\varphi}(v_\varrho, v_\varphi)\nabla v_\varrho - \frac{v_\varphi^2}{2}\frac{\partial}{\partial\varphi}\Gamma^{\varrho}_{\varphi\varphi}(v_\varrho, v_\varphi)\nabla v_\varphi,$$

the first estimate of Lemma 27, (84), (86) and (93) imply

$$\Big| |\nabla v_\rho - \nabla \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} - |\nabla v_\Upsilon - \nabla \hat{l}_\rho|_{L^\infty(B_{\frac{15\vartheta R}{16}}(0))} \Big| \le C\varphi_0^2. \tag{109}$$

Combined with the mean value inequality, we therefore conclude as in (38) that for $r \in (0, \frac{R}{2}]$,

$$\sup_{B_{r\vartheta}(0)} |v_\rho - \hat{l}_\rho| \le C r^{1+\alpha} \sup_{B_{R\vartheta}(0)} |v_\rho - \hat{l}_\rho| + C r \vartheta \varphi_0^2. \tag{110}$$

Exactly as in (39), we have

$$\sup_{B_{\vartheta R}(0)} |v_\rho - \hat{l}_\rho| \le C\Big(\sup_{B_{\vartheta R}(0)} |v_\rho - L_\rho| + \vartheta\varphi_0^2\Big). \tag{111}$$

The above two inequalities imply

$$\sup_{B_{\vartheta r}(0)} |v_\rho - \hat{l}_\rho| \le C r^{1+\alpha} \sup_{B_{\vartheta R}(0)} |v_\rho - L_\rho| + C r \vartheta \varphi_0^2 \quad \text{for all } r \in \Big(0, \frac{R}{2}\Big].$$

Since $v_\varphi(x) = \hat{l}_\varphi(x)$, this estimate and Lemma 4 imply the lemma. □

7. Appendix

In this section, we prove some lemmas regarding blow-up maps and the order function. In a slight different language and for two-dimensional domains these results first appeared in [20], but we are including them here for the sake of completeness. We first need the following definition.

Definition 30. We say that a map $v_0 : B_1(0) \to Y_0$ into an NPC space is *piecewise a function* if, for any connected component Ω_0 of the set $\{x \in B_1(0) : v_0(x) \neq v_0(0)\}$, the pullback distance function of $v_0|_{\Omega_0}$ is equal to the pullback distance function of the function $f := d(v_0, v_0(0))|_{\Omega_0}$.

Lemma 31. *There exists $\epsilon_0 > 0$ depending only on the dimension of the domain such that if a homogeneous harmonic map $v_0 : B_1(0) \to Y_0$ is piecewise a function, then*

$$\mathrm{Ord}^{v_0}(0) = 1 \quad or \quad \mathrm{Ord}^{v_0}(0) \ge 1 + \epsilon_0.$$

Moreover, if $\mathrm{Ord}^{v_0}(0) = 1$, *then the pullback distance function of v_0 is equal to that of a linear function.*

Proof. Let Ω_0 be a connected component of the set $\{x \in B_r(0) : v_0(x) \neq v_0(0)\}$ and let $f = d(v_0, v_0(0))|_{\Omega_0}$. We claim that

$$v_0|_{\Omega_0} \text{ maps into a geodesic in } Y_0. \tag{112}$$

Indeed, for $r \in (0,1)$, let $x_{\Omega_0} \in \overline{\Omega_0 \cap B_r(0)}$ such that

$$d(v_0(x_{\Omega_0}), v_0(0)) = \max_{x \in \overline{\Omega_0 \cap B_r(0)}} d(v_0(x), v_0(0)).$$

For $x_0 \in \Omega_0 \cap \overline{B_r(0)}$,

$$\begin{aligned} d(v_0(x_0), v_0(0)) &= \lim_{x \to 0} d(v_0(x_0), v_0(x)) = f(x_0) - \lim_{x \to 0} f(x) = f(x_0), \\ d(v_0(x_{\Omega_0}), v_0(x_0)) &= f(x_{\Omega_0}) - f(x_0), \\ d(v_0(x_{\Omega_0}), v_0(0)) &= \lim_{x \to 0} d(v_0(x_{\Omega_0}), v_0(x)) = f(x_{\Omega_0}) - \lim_{x \to 0} f(x) = f(x_{\Omega_0}). \end{aligned}$$

Thus,

$$d(v_0(x_{\Omega_0}), v(0)) = d(v_0(x_{\Omega_0}), v_0(x_0)) + d(v_0(x_0), v_0(0)),$$

which implies $v_0(x_0)$ lies on a geodesic from $v_0(0)$ and $v_0(x_{\Omega_0})$ and proves (112).

We thus may assume that $v_0\big|_{\Omega_0} = f$, hence the harmonicity and homogeneity of v_0 imply

$$\triangle f = 0 \quad \text{and} \quad f(r,\theta) = r^\alpha F(\theta) \quad \text{in } \Omega_0,$$

where $\alpha = \mathrm{Ord}^{v_0}(0)$, $F : \Lambda := \Omega_0 \cap \partial B_1(0) \subset S^{n-1} \to \mathbf{R}^+$ and $\theta = (\theta^1, \dots, \theta^{n-1})$ are the coordinates of S^{n-1}. Combining the above two equations, we conclude that F is a Dirichlet eigenfunction satisfying the following equation in the domain $\Lambda \subset S^{n-1}$:

$$\alpha(\alpha + n - 1)F + \triangle_\theta F = 0$$

The Faber–Krahn theorem implies that the volume of Λ, and hence of Ω_0, has a lower bound depending on α and n. In particular, we conclude that there exists $\epsilon_0 \in (0,1]$ sufficiently small depending only on the domain dimension n such that if $\mathrm{Ord}^{v_0}(0) < 1 + \epsilon_0$, then there exists at most two connected components of $\{x \in B_r(0) : v_0(x) \neq v_0(0)\}$. The maximum principle applied to the subharmonic function $f = d(v_0, v_0(0))$ imply that there cannot be only one component. Therefore, there exist exactly two connected components Ω_+ and Ω_- of the set $\{x \in B_r(0) : v_0(x) \neq v_0(0)\}$ which v_0 maps into two geodesics γ_+ and γ_- by (112). Since v_0 is harmonic, $\gamma_+ \cup \{v_0(0)\} \cup \gamma_-$ is a geodesic. Thus, the function

$$\hat{f}(x) = \begin{cases} d(v_0(x), v_0(0)), & x \in \Omega_+, \\ -d(v_0(x), v_0(0)), & x \in \Omega_-, \\ 0, & \text{otherwise}, \end{cases}$$

is a harmonic function which agrees with the pullback distance function of v_0. Since

$$\mathrm{Ord}^{\hat{f}}(0) = \mathrm{Ord}^{v_0}(0) < 1 + \epsilon_0 < 2$$

and the order of a harmonic function is integer valued, we conclude

$$\mathrm{Ord}^{v_0}(0) = \mathrm{Ord}^{\hat{f}}(0) = 1. \qquad \square$$

The following proposition is motivated by the fact that tangent maps into $\overline{\mathbf{H}}$ or $\overline{\mathbf{H}}_2$ are piecewise a functions as will be shown in Lemma 34.

Proposition 32. *If a homogeneous harmonic map $v_0 : B_1(0) \to Y_0$ is piecewise a function, then*

$$\dim_{\mathcal{H}}(\mathcal{S}_0(v_0)) \le n-2,$$

where $\mathcal{S}_0(v_0) = \{x \in B_1(0) : \mathrm{Ord}^{v_0}(x) > 1\}$.

Proof. Assume that $\mathcal{H}^s(\mathcal{S}_0(v_0)) > 0$ for $s \in [0,n]$. By [11], we can choose $x_0 \in \mathcal{S}_0(v_0)$ such that $x_0 \neq 0$ and

$$\lim_{\sigma\to 0} \mathcal{H}^s(\mathcal{S}_0(v_{0\sigma})) = \lim_{\sigma\to 0} \frac{\mathcal{H}^s(\mathcal{S}_0(v_0)) \cap B_\sigma(x_0))}{\sigma^s} \ge 2^{-s},$$

where $v_{0\sigma}$ is the blow-up map of v_0 at x_0. By rotating if necessary, we can assume that $x_0 = (0,\dots,0,|x_0|)$. By Lemma 31, $\mathrm{Ord}^{v_0}(x_0) \ge 1+\epsilon_0$. The homogeneity of v_0 implies that $\mathrm{Ord}^{v_0}(tx_0) \ge 1+\epsilon_0$ for all $t > 0$, and hence we have $\mathrm{Ord}^{v_{0\sigma}}(0,\dots,0,t) \ge 1+\epsilon_0$ for all $t > 0$. In turn, this implies $\mathrm{Ord}^{v_{0*}}(0,\dots,0,t) \ge 1+\epsilon_0$ for all $t > 0$ where v_{0*} is a tangent map of v_0 at x_0.

We claim that v_0 satisfies the order gap property in the sense that there exists $\epsilon_0 > 0$ such that for any $x \in \mathcal{S}$, either $\mathrm{Ord}^v(x) = 1$ or $\mathrm{Ord}^v(x) \ge 1+\epsilon_0$. Indeed, for $x \in B_1(0)\backslash v_0^{-1}(v_0(0))$ and $r > 0$ sufficiently small, we can assume that $v_0|_{B_r(x)}$ is function. Thus, $\mathrm{Ord}^{v_0}(x) = 1$ or $\mathrm{Ord}^{v_0}(x) \ge 2$. For $x \in v_0^{-1}(v_0(0))$ and $r > 0$ sufficiently small, $v_0|_{B_r(x)}$ (after identifying $x = 0$) is piecewise a function. Thus, a tangent map of v_0 is piecewise a function and applying Lemma 31 to it, we can conclude $\mathrm{Ord}^{v_0}(x) = 1$ or $\mathrm{Ord}^{v_0}(x) \ge 1+\epsilon_0$. We have proved that v_0 has the order gap property, and hence [7, Lemma 76] implies that for the blow-up map v_{0*} of v_0 at x_0

$$\dim_{\mathcal{H}}(\mathcal{S}_0(v_{0*})) \ge s.$$

Moreover, v_{0*} is independent of a direction, and the restriction of v_{0*} to $\mathbb{R}^{n-1}$ spanned by the first $(n-1)$ standard basis vectors, denoted $\tilde{v}_{0*}$, is a homogeneous map of degree $\alpha_{0*} \ge 1+\epsilon_0$. We then have

$$\mathcal{S}_0(v_{0*}) = \mathcal{S}_0(\tilde{v}_{0*}) \times \mathbb{R} \quad \text{and} \quad \dim_{\mathcal{H}}(\mathcal{S}_0(\tilde{v}_{0*})) \ge s-1.$$

Furthermore, since v_0 is piecewise a function, the blow-up maps $\{v_{0\sigma}\}$ are also piecewise a function. Since $\{v_{0\sigma}\}$ converges to $\tilde{v}_{0*}$ locally uniformly in the pullback sense, $\tilde{v}_{0*}$ is piecewise a function. Thus, if $s > n-2$, we may repeat this argument inductively to produce a geodesic with order not equal to 1 for some point, a contradiction. This shows $s \le n-2$. □

Lemma 33. *Let $v_0 : B_1(0) \to Y_0$ be a homogeneous harmonic map into an NPC space. If there exists a sequence of harmonic maps $\{w_i = (w_i^\rho, w_i^\phi) : B_1(0) \to Y\}$, where $Y = \overline{\mathbf{H}}$ or $\overline{\mathbf{H}}_2$ with uniformly bounded energy converging locally uniformly in the pullback sense to v_0 and $\lim_{i\to\infty} w_i(0) = P_0$, then v_0 is piecewise a function. In particular,*

$$\mathrm{Ord}^{v_0}(0) = 1 \quad \textit{or} \quad \mathrm{Ord}^{v_0}(0) \ge 1+\epsilon_0.$$

and

$$\dim_{\mathcal{H}}(\mathcal{S}_0(v_0)) \le n-2.$$

Moreover, if $\mathrm{Ord}^{v_0}(0) = 1$, then the pullback distance function of v_0 is equal to that of a linear function.

Proof. Since $E^{w_i}(1)$ is uniformly bounded, [15, Theorem 2.4.6] implies that, for any $r \in (0,1)$, there exists $C > 0$ such that for all i and $x \in B_r(0) \backslash \{x : w_i(x) \neq P_0\}$,

$$|\nabla w_i^\rho|(x) \leq C, \quad (w_i^\rho(x))^3 |\nabla w_i^\phi|(x) \leq C. \tag{113}$$

Let Ω_0 be a connected component of $B_1(0) \backslash v_0^{-1}(v_0(0))$ and $f : \Omega_0 \to \mathbb{R}^+$ as in Definition 30. Fix $x_{\Omega_0} \in \Omega_0$ and let K be an arbitrary domain compactly contained in Ω_0 and with $x_{\Omega_0} \in K$. By the local uniform convergence in the pullback sense of w_i to v_0 and the fact that $\lim_{i \to \infty} w_i(0) = P_0$, we also have the local uniform convergence of w_i^ρ to the function f. Thus, the function w_i^ρ is bounded away from 0 in K for i sufficiently large. Therefore (113) implies that w_i^ϕ is uniformly Lipschitz in K, and there exists a subsequence of $\{w_i^\phi - w_i^\phi(x_{\Omega_0})\}$ for some fixed x_{Ω_0} (which we shall still denote by $\{w_i^\phi - w_i^\phi(x_{\Omega_0})\}$ by an abuse of notation) that converges uniformly in K. By taking a compact exhaustion of Ω_0 by a sequence of sets possessing the properties of K and applying a diagonalization procedure, we can assume (by taking a subsequence if necessary) that $\{w_i^\phi - w_i^\phi(x_{\Omega_0})\}$ converges locally uniformly to some function g in Ω_0. Thus, $\{(w_i^\rho, w_i^\phi - w_i^\phi(x_{\Omega_0}))\}$ converges locally uniformly in Ω_0 to the map $(f,g) : \Omega_0 \to \mathbf{H}$. Since $\{(w_i^\rho, w_i^\phi - w_i^\phi(x_{\Omega_0}))\}$ is a sequence of harmonic maps into a smooth Riemannian manifold $(\mathbf{H}, g_{\mathbf{H}})$, this convergence is actually locally C^k (for any k). The map w_i is harmonic which implies that the functions w_i^ρ and w_i^ϕ satisfy

$$w_i^\rho \triangle w_i^\rho = 3(w_i^\rho)^6 |\nabla w_i^\phi|^2 \quad \text{in } \Omega_0.$$

Thus, the functions f and g also satisfy

$$f \triangle f = 3 f^6 |\nabla g|^2 \quad \text{in } \Omega_0. \tag{114}$$

Furthermore, the homogeneity of v_0 implies the homogeneity of f. Thus, Ω_0 is a cone and we can write

$$f(r,\theta) = r^\alpha F(\theta) \quad \text{in } \Omega_0,$$

where α is the degree of homogeneity of v_0, $F : \Lambda := \Omega_0 \cap \partial B_1(0) \subset S^{n-1} \to \overline{\mathbb{R}}_+$ and $\theta = (\theta^1, \dots, \theta^{n-1})$ are the coordinates of S^{n-1}. The above two equations imply that

$$\alpha(\alpha + n - 1)F + \triangle_\theta F = r^{4\alpha+2} F^6(\theta) |\nabla g|^2.$$

Since this inequality holds for any $r > 0$, we can conclude that $|\nabla g|^2 = 0$. Since

$$w_i^\phi - w_i^\phi(x_{\Omega_0}) = 0$$

at $x = x_{\Omega_0}$, we see that $g(x_{\Omega_0}) = 0$. Hence this implies $g = 0$ in Ω_0 and $(w_i^\rho, w_i^\phi - w_i^\phi(x_{\Omega_0}))$ converges locally uniformly to $(f, 0)$ in Ω_0. In particular, we conclude that v_0 is piecewise a function. The rest of the assertions follow from Lemma 31 and Proposition 32. □

Lemma 34. *If $u = (u^\rho, u^\phi) : B_1(0) \to Y$ is a harmonic map, where $Y = \overline{\mathbf{H}}$ or $\overline{\mathbf{H}}_2$, $x \in B_1(0)$ and u_* is a tangent map of u at x, then*

$$\mathrm{Ord}^{u_*}(0) = 1 \quad or \quad \mathrm{Ord}^{u_*}(0) \geq 1 + \epsilon_0, \quad \dim_{\mathcal{H}}(\mathcal{S}_0(u_*)) \leq n - 2.$$

Moreover, if $u(x) = P_0$ and $\mathrm{Ord}^u(x) = 1$, then the pullback distance function of a tangent map u_* of u at x is equal to that of a linear function.

Proof. Let $x \in B_1(0)$. Let $\{u_{\sigma_i} = (u^{\rho}_{\sigma_i}, u^{\phi}_{\sigma_i})\}$ and u_* be the blow-up maps and a tangent map of u at x such that u_{σ_i} converges to u_* locally uniformly in the pullback sense. First, assume $u(x) \neq P_0$. Then u maps a neighborhood of x into a smooth Riemannian manifold, and u_* maps into $T_{u(x)}\mathbf{H} = \mathbb{R}^2$. Therefore, if $u(x) \neq P_0$, we conclude the assertion of the lemma with $\epsilon_0 = 1$. Next, assume $u(x) = P_0$ which then implies $u_{\sigma_i}(0) = P_0$. We obtain the assertion of the lemma by applying Lemma 33 with $w_i = u_{\sigma_i}$ and $v_0 = u_*$. □

We now arrive at the following.

Theorem 35. *If $u = (u^\rho, u^\phi) : B_1(0) \to Y$ is a harmonic map, where $Y = \overline{\mathbf{H}}$ or $\overline{\mathbf{H}}_2$, then the set of higher order points of u is of Hausdorff codimension at least 2, i.e.*

$$\dim_{\mathcal{H}}(\mathcal{S}_0(u)) \leq n - 2.$$

Proof. This follows immediately from [7, Theorem 78] and Lemma 34. □

If $\mathrm{Ord}^{v_0}(0) = 1$ in Lemma 33, then we can assume (after rotating if necessary) that $v_0(x) = Ax^1$ for some constant A. The next lemma shows that we can embed the image of v_0 in $\overline{\mathbf{H}}$ or in $\overline{\mathbf{H}}_2$ so that the convergence takes place in $\overline{\mathbf{H}}$ or in $\overline{\mathbf{H}}_2$ (instead of the convergence in the pullback sense).

Lemma 36. *Define $B_r^+(0) = \{x \in B_r(0) : x^1 \geq 0\}$ and assume that the sequence $u_i : B_1^+(0) \to \overline{\mathbf{H}}$ with $u_i(0) = P_0$ converges locally uniformly in the pullback sense to a function $f(x) = Ax^1$. Then there exists a sequence of maps $l_i^+ : B_1^+(0) \to \overline{\mathbf{H}}$ defined by*

$$l_i^+(x) = \begin{cases} (Ax^1, \phi_i^+), & x^1 > 0, \\ P_0, & x^1 = 0, \end{cases}$$

for a constant $A > 0$ and sequences $\{\phi_i^+\}$ such that, for any $r \in (0,1)$,

$$\lim_{i \to \infty} \sup_{B_r^+(0)} d(u_i, l_i^+) = 0$$

Proof. Choose base points $x_+ = (\frac{1}{2}, 0, \dots, 0) \in B_1^+(0)$. Define

$$f_i = (f_{i\rho}, f_{i\phi}) : B_1(0) \to \overline{\mathbf{H}}$$

by setting

$$f_{i\rho}(x) = u_{i\rho}(x), \quad f_{i\phi}(x) = u_{i\phi}(x) - u_{i\phi}(x^+).$$

Since the translation $(\rho, \phi) \mapsto (\rho, \phi - \phi_0)$ is an isometry of $\overline{\mathbf{H}}$, it suffices to prove

$$d(f_i, l^0) \to 0 \quad \text{uniformly in } \overline{B}_r^+(0), \tag{115}$$

where $l^0 : B_r(0) \to \overline{\mathbf{H}}$ is defined by

$$l^0(x) = \begin{cases} (Ax^1, 0) \in \mathbf{H}, & x^1 > 0, \\ P_0, & x^1 = 0. \end{cases} \tag{116}$$

Again, by the isometry above, we have $d(f_i, f_i) = d(u_i, u_i)$, and hence f_i converges uniformly to u_* in the pullback sense; i.e. uniformly in $\overline{B_r^+}(0)$,

$$d(f_i(\cdot), f_i(\cdot)) \to d(u_*(\cdot), u_*(\cdot)) = d(l^0(\cdot), l^0(\cdot)). \tag{117}$$

Since

$$f_{i\rho}(x_+) = u_{i\rho}(x_+) = d(u_i(x_+), P_0) = d(u_i(x_+), u_i(0)),$$

we have by (117) that

$$\lim_{i\to\infty} f_{i\rho}(x_+) = d(u_*(x_+), u_*(0)) = Ax_+^1 = l^0(x_+).$$

Therefore, f_i is normalized so that

$$f_{i\phi}(x_+) = 0 \quad \text{and} \quad \lim_{i\to\infty} d(f_i(x_+), l^0(x_+)) = 0. \tag{118}$$

Claim 1. *For each $i \in \mathbb{N}$, there exists a subsequence of $\{f_j\}$ (which we call $\{f_j\}$ by an abuse of notation) that converges uniformly to l^0 on $\{x \in B_r(0) : x^1 \geq \frac{1}{i}\}$.*

Proof. Since the sequence f_j is uniformly Lipschitz continuous, the claim follows from Arzela–Ascoli by proving the following statement: there exist $\phi_0 > 0$ and $i_0 > 0$ such that f_j maps $B(i) := \{x \in B_r(0) : x^1 \geq \frac{1}{i}\}$ to a compact set

$$K := \{(\rho, \phi) : 0 \leq \rho \leq 2A,\ -\phi_0 \leq \phi \leq \phi_0\}$$

for all $j \geq i_0$. If not, then there exists a sequence $x_{j'} \in \{x \in B_r(0) : x^1 \geq \frac{1}{i}\}$ such that $x_{j'} \to x$ and $f_{j'\phi}(x_{j'}) \to \pm\infty$. Now note that by (117), for sufficiently large j and for all $x \in \overline{B_r^+}(0)$, we have

$$0 \leq f_{j\rho}(x) = d(f_j(x), P_0) = d(f_j(x), f_j(0)) < A + d(l^0(x), l^0(0)) \leq 2A.$$

On the one hand, again by (117) and (118),

$$\begin{aligned}\lim_{j'\to\infty} d(f_{j'}(x_{j'}), l^0(x_+)) &\leq \lim_{j'\to\infty} [d(f_{j'}(x_{j'}), f_{j'}(x_+)) + d(f_{j'}(x_+), l^0(x_+))] \\ &= d(l^0(x), l^0(x_+)) \leq A\left(\frac{1}{2} - \frac{1}{i}\right) < \frac{A}{2}.\end{aligned}$$

Furthermore, $f_{j'\phi}(x_{j'}) \to \pm\infty$ implies

$$\begin{aligned}\liminf_{j'\to\infty} d(f_{j'}(x_{j'}), l^0(x_+)) &= \liminf_{j'\to\infty} d((f_{j'\rho}(x_{j'}), f_{j'\phi}(x_{j'})), l^0(x_+)) \\ &= \liminf_{j'\to\infty} f_{j'\rho}(x_{j'}) + \frac{A}{2} \geq \frac{A}{2}.\end{aligned}$$

This is a contradiction. □

Claim 2. *There exists a subsequence of $\{f_i\}$ (denoted by $\{f_i\}$ by an abuse of notation) such that*

$$d(f_i(x), l^0(x)) < \frac{1}{i} \quad \text{for } x^1 \geq \frac{1}{i}$$

and

$$|d(f_i(x), f_i(0)) - d(l^0(x), l^0(0))| < \frac{1}{i} \quad \text{for all } x \in \overline{B_r^+}(0).$$

Proof. By (117), we can choose j_1 sufficiently large such that

$$|d(f_{j_1}(x), f_{j_1}(0)) - d(l^0(x), l^0(0))| < \frac{1}{2} \quad \text{for all } x \in \overline{B}_1^+(0).$$

Assume $j_1, \dots, j_{i-1}$ are defined. Now let j_i be sufficiently large such that $j_i > j_{i-1}$,

$$d(f_{j_i}(x), l^0(x)) < \frac{1}{i} \quad \text{for } x^1 \geq \frac{1}{i}$$

and

$$|d(f_{j_i}(x), f_{j_i}(0)) - d(l^0(x), l^0(0))| < \frac{1}{i} \quad \text{for all } x \in \overline{B}_r^+(0).$$

The existence of such j_i is guaranteed by Claim 1 and (117). The subsequence $\{f^{j_i}\}$ thus inductively defined satisfies the assertion of the lemma. □

For $0 \leq x^1 \leq \frac{1}{i}$, we have by the triangle inequality and Claim 2 that

$$\begin{aligned} d(f_i(x), l^0(x)) &\leq d(f_i(x), P_0) + d(l^0(x), P_0) \\ &= d(f_i(x), f_i(0)) + d(l^0(x), P_0) \\ &\leq d(l^0(x), l^0(0)) + \frac{1}{i} + d(l^0(x), P_0) \\ &= \frac{1}{i} + 2d(l^0(x), P_0) \\ &\leq \frac{1+2A}{i}. \end{aligned}$$

By Claim 1, this completes the proof of (115) and hence of the lemma. □

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