Harmonic maps between singular spaces I

Georgios Daskalopoulos[1]
Brown University
daskal@math.brown.edu

Chikako Mese[2]
Johns Hopkins University
cmese@math.jhu.edu

Abstract

We discuss regularity questions for harmonic maps from a n-dimensional Riemannian polyhedral complex X to a non-positively curved metric space. The main theorems assert, assuming Lipschitz regularity of the metric on the domain complex, that such maps are locally Hölder continuous with explicit bounds of the Hölder constant and exponent on the energy of the map and the geometry of the domain and locally Lipschitz continuous away from the $(n-2)$-skeleton of the complex. Moreover, if x is a point on the k-skeleton ($k \leq n-2$) we give explicit dependence of the Hölder exponent at a point near x on the combinatorial and geometric information of the link of x in X and the link of the k-dimensional skeleton in X at x.

1 Introduction

The seminal work of M. Gromov and R. Schoen [GS] extends the study of harmonic maps between smooth manifolds to the case when the target is a Riemannian simplicial complex of non-positive curvature. The theory of harmonic maps into singular spaces was expanded substantially by the work of N. Korevaar, R. Schoen [KS1], [KS2] and J. Jost [J] where they consider targets that are arbitrary metric spaces of non-positive curvature. (Such spaces are called NPC or CAT(0) if they are simply connected.) One important motivation for considering singular spaces in the theory of harmonic maps is in studying group representations. The main application of the Gromov-Schoen theory is to establish a certain case of non-Archemedean superrigidity complementing Corlette's Archemedean superrigidity for lattices in groups of real rank 1 [Co].

The next step in the generalization of the harmonic map theory is to replace smooth domains by singular ones. This problem is also motivated by superrigidity, in this case when the domain group is non-Archemedean. The consideration of a Riemannian simplicial complex as the domain space for harmonic maps

[1]supported by research grant NSF DMS-0604930
[2]supported by research grant NSF DMS-0450083

seems to have been initiated by J. Chen [Ch]. Subsequently, this theory was further elaborated by J. Eells and B. Fuglede [EF] and [F]. In particular, they show Hölder continuity for harmonic maps under an appropriate smoothness asumption for the metric on each simplex.

The development of the harmonic map theory from a Riemannian complex is important in the study of non-Archedemean lattices. Considering a 2-dimensional domains, [DM1], [DM2] and [DM3] establish fixed point and rigidity theorems of harmonic maps from certain flat 2-complexes. The key issue in the techniques introduced in these papers is to prove regularity theorems strong enough to be able to apply differential geometric methods.

Recall that the main idea of [GS] is also to show that harmonic maps are regular enough so that Bochner methods could be used in the setting of singular targets. In particular, the fundamental regularity result of [GS] and [KS1] is that harmonic maps from a smooth Riemannian domain into an NPC target are locally Lipschitz continuous. As noted in [Ch], this statement no longer holds when we replace the domain by a polyhedral space. On the other hand, we have found in [DM2] and [DM3] that modulus of continuity better than Hölder is crucial in applications. This necessitates stronger regularity results than Hölder.

This paper is meant to be the state of the art in the regularity theory of harmonic maps from Riemannian cell complexes to non-positively curved metric spaces (cf. Section 2 for precise definitions). Our first theorem concerns Hölder continuity of harmonic maps. This is a generalization of the result of [EF] for Lipschitz metrics.

Theorem (cf. Theorem 42) *Let $B(r)$ be a ball or radius r around a point in an admissible complex X endowed with a Lipschitz Riemannian metric g and (Y,d) an NPC space. If $f:(B(r),g)\to(Y,d)$ is a harmonic map, then there exist C and $\gamma>0$ so that*

$$d(f(x),f(y))\le C|x-y|^{\gamma}\quad \forall x,y\in B(\varrho r).$$

Here, C and γ only depend on the total energy E^f of the map f, $(B(r),g)$ and $\varrho\in(0,1)$.

Note that our approach to Hölder continuity follows the one in [GS] and [Ch] and is completely different from the one in [EF] and [F]. In our case, a variant of the Gromov-Schoen monotonicity formula allows us to obtain energy decay estimates which in turns imply Hölder continuity by an adaptation of an argument due to Morrey. The technical difficulty is that we make no assumption that the boundary of each simplex is totally geodesic as it is implicitly assumed in [Ch]. Our method also differs from the one in [GS] due to the fact that for singular domains the monotonicity formula does not hold for large balls (cf. Remark 3.3 of [DM1]. The main technical hurdle is to obtain energy decay estimates with uniform radius and this is handled in Section 4.

Our main theorem concerns better regularity of harmonic maps. More precisely, we show that harmonic maps are Lipschitz continuous away from the codimension 2 skeleton $X^{(n-2)}$ of X. For points that lie on the lower dimensional skeleta, we also give an estimate of the Hölder exponent of the harmonic map in terms of the first eigenvalue of the link of the normal stratum of the skeleton. More precisely, let $x \in X^{(k)} - X^{(k-1)}$ and let $N = N(x)$ denote the link of $X^{(k)}$ at x along with the metric induced by the given Lipschitz Riemannian metric on X. Note that N is a spherical $(n-k-1)$-complex. Set

$$\lambda_1^N := \inf_{Q \in Y} \lambda_1(N, T_Q Y),$$

where $\lambda_1(N, T_Q Y)$ denotes the first eigenvalue of the Laplacian of N with values in the tangent cone of Y at Q (See Section 8 for further details). More precisely, our main theorem is as follows:

Main Theorem (cf. Theorem 76) *Let $B(r)$ be a ball of radius r around a point x in an admissible complex X endowed with a Lipschitz Riemannian metric g, (Y, d) an NPC space and $f : (B(r), g) \to (Y, d)$ a harmonic map.*

(1) If $x \in X - X^{(n-2)}$, let d denote the distance of x to $X^{(n-2)}$. Then for $\varrho \in (0, 1)$ and $d' \leq \min\{\varrho r, \varrho d\}$, f is Lipschitz continuous in $B(d')$ with Lipschitz constant depending on the total energy E^f of the map f, $(B(r), g)$, d and ϱ.

(2) If $x \in X^{(k)} - X^{(k-1)}$ for $k = 0, ..., n-2$, let d denote the distance of x to $X^{(k-1)}$. Then for $\varrho \in (0, 1)$ and $d' \leq \min\{\varrho r, \varrho d\}$, f is Hölder continuous in $B(d')$ with Hölder exponent and constant depending on on the total energy E^f of the map f, $(B(r), g)$, d and ϱ. More precisely, the Hölder exponent α has a lower bound given by the following: If $\lambda_1^N \geq \beta (> \beta)$ then $\alpha(\alpha + n - k - 2) \geq \beta (> \beta)$. In particular, if $\lambda_1^N \geq n-k-1$, then f is Lipschitz continuous in a neighborhood of x.

The paper is organized as follows: In Section 2 we define our domain and target spaces and recall the notion of harmonic maps. In Section 3 we prove the monotonicity formula in our setting and in Section 4 we discuss the Hölder continuity of harmonic maps. Section 3 is in some sense the heart of the paper as all subsequent results depend on it. Though similar in spirit with the monotonicity formula of [GS] it also differs significantly in the fact that we show that the relevant quantity (the order function) is not monotone as in [GS]. Nonetheless, we show that the order function has a well defined limit. This is necessary due to the fact that the different strata in the complex are not assumed to be totally geodesic. As mentioned before, in Section 4 we include a proof of the Hölder continuity of harmonic maps with a slightly more relaxed assumption on the metric of the complex than in [EF] and [F]. The purpose of Sections 5 and 6 is to construct a tangent map. We then establish properties of maps from a flat domain in order to analytize the tangent map in section 7. Finally, Section 8 is devoted to the proof of the Main Theorem.

Notation Throughout the paper, unless mentioned otherwise (Y, d) or simply Y will denote an NPC metric space.

2 Domain and target spaces

2.1 Local models

We now introduce our local models which will represent a neighborhood of a point in a complex. A half space is a connected component H of $\mathbf{R}^n - h$ where h is an affine hyperplane. By a normalized half space, we will mean a half space H so that the hyperplane h that defines H contains the origin $\vec{0}$. We say the normalized half spaces $H_1, ..., H_\nu$ are linearly independent if the normals to the hyperplanes $h_1, ..., h_\nu$ defining the half spaces are linearly independent. A wedge (or a n-dimensional ν-wedge) W is the closure of the intersection of ν number of linearly independent normalized half spaces $H_1, ..., H_\nu$. By its construction, every wedge is a n-dimensional cone in $\mathbf{R}^n$ with $\vec{0}$ as the vertex. Wedge angles are the angles between any pair of vectors $h_1, ..., h_\nu$. In particular a 2-wedge has one wedge angle, and in general a ν-wedge has $\frac{\nu(\nu-1)}{2}$ number of wedge angles.

A face of the wedge is an intersection $W \cap h_{i_1} \cap ... \cap h_{i_j}$, $1 \leq i_1 \leq ... \leq i_j \leq \nu$. For example, the intersection $W \cap h_1$ is a face which is a $(n-1)$-dimensional linear subspace of $\mathbf{R}^n$ and the intersection $W \cap h_1 \cap ... \cap h_\nu$ is a face which is a $(n-\nu)$-dimensional linear subspace of $\mathbf{R}^n$. This latter face is the lowest dimensional face of W and we denote it by D. We will use the coordinates of $\mathbf{R}^n$ to label points in W. For simplicity, we always choose the coordinate system $(x^1, ..., x^n)$ of $\mathbf{R}^n$ so that D is given as $x^{n-\nu+1} = ... = x^n = 0$.

Let $W_1, \ldots, W_l$ be n-dimensional ν-wedges and let $\{F_i^a\}_{a=1,\ldots,\nu}$ be the set of all $(n-1)$-dimensional faces of W_i for $i = 1, \ldots, l$. For any $i, j = 1, \ldots l$ with $i \neq j$ let $\varphi_{ij}^{ab} : F_i^a \to F_j^b$ be a possibly empty linear isometry called a *gluing map* of F_i^a and F_j^b and let Φ_{ij} be a set of all gluing maps $\varphi_{ij}^{ab} : F_i^a \to F_j^b$ for $a, b = 1, \ldots, \nu$. Let Φ be the union of Φ_{ij} for $i, j = 1, \ldots, l$ and $i \neq j$.

Definition 1 *A dimension-n, codimension-ν local model $\mathbf{B} = \cup W_i / \sim$ is a disjoint union of n-dimensional ν-wedges $W_1, \ldots, W_l$ along with an equivalence relation $\sim$ defined by setting $x \sim x'$ if $\varphi(x) = x'$ for $\phi \in \Phi$. We further require (i) the cardinality of each Φ_{ij} is at most 1 and (ii) for every $i = 1, \ldots, l$ and $a = 1, \ldots, \nu$, there exists a nonempty gluing map in Φ with F_i^a as a domain or target.*

When we have two n-dimensional 1-wedges, i.e. two half spaces, glued together along $D = \{x^n = 0\}$, the local model is simply $\mathbf{R}^n$ and this will be referred to as a codimension-0 local model. Given a face F of a wedge W_i, we will also call its equivalence class in $\mathbf{B}$ a *face*. The boundary of a local model is the union of all $(n-1)$-dimensional faces which belong to exactly one wedge. Throughout the paper we also assume that our local models have empty boundary. Also note that property (iii) implies that our local models $\mathbf{B}$ are connected and admissible i.e that $\mathbf{B} - F$ is connected for any $(n-2)$-dimensional face F.

Since W is a subset of $\mathbf{R}^n$, there is a natural Euclidean metric inherited from $\mathbf{R}^n$. This defines an Euclidean metric δ on $\mathbf{B}$. For $x, y \in \mathbf{B}$, let $|x-y|$ be the induced distance function from δ. Set $\mathbf{B}(r)$ be the r-ball centered at the origin of $\mathbf{B}$ and $W(r) = \mathbf{B}(r) \cap W$ for any wedge W of $\mathbf{B}$. For the sake of simplicity, we will also refer to $W(r)$ as a wedge (of $\mathbf{B}(r)$). Also for $x \in \mathbf{B}$ we will denote by $B_x(r)$ the Eucliden r-ball around x. Note that *throughout the paper all balls will be taken with respect to the metric* δ.

We now give examples of wedges in dimension 2 and dimension 3. (i) The only two dimensional 1-wedge (up to linear isometry) is the half plane $\{(x,y) \in \mathbf{R}^2 : y \geq 0\}$. We consider a model space $\mathbf{B}$ where k copies of 1-wedges are glued together along $D = \{(x,y) \in \mathbf{R}^2 : y = 0\}$. This example models a neighborhood of an edge point of a two dimensional simplicial complex. (ii) An example of a two dimensional 2-wedge is the first quadrant $\{(x,y) \in \mathbf{R}^2 : x, y \geq 0\}$. Another example is the set $W = \{(x,y) \in \mathbf{R}^2 : \sqrt{3}x \geq y \geq 0\}$. A vertex point of a two dimensional simplicial complex can be modelled by a model space where l copies of W are glued together along their faces (in this case lines $y = 0$ or $y = \sqrt{3}x$) according to the combinatorial information of the complex. Note that D is this case is the point $x = y = 0$. (iii) The only three dimensional 1-wedge (up to linear isometry) is the half space $\{(x,y,z) \in \mathbf{R}^3 : z \geq 0\}$. The model space $\mathbf{B}$ where l copies of 1-wedges are glued together along $D = \{(x,y,z) \in \mathbf{R}^3 : z = 0\}$ models a neighborhood of the 2-skeleton in a three dimensional simplicial complex. (iv) An example of a three dimensional 2-wedge is $\{(x,y,z) \in \mathbf{R}^3 : y, z \geq 0\}$. Another example is the set $W = \{(x,y,z) \in \mathbf{R}^3 : \sqrt{3}y \geq z \geq 0\}$. A neighborhood of a point on a 1-skeleton of a three dimensional simplicial complex can be modelled by a model space $\mathbf{B}$ where l copies of W are glued together along their faces according to the combinatorial information. Here, $D = \{(x,y,z) \in \mathbf{R}^3 : y = z = 0\}$. (v) An example of a three dimensional 3-wedge is the first octant $\{(x,y,z) \in \mathbf{R}^3 : x, y, z \geq 0\}$. Another example is the set W consisting of points of the form $\sum_{i=1}^3 t_i v_i$, $t_i \geq 0$ where $v_1 = (1,0,0)$, $v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $v_3 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}})$. Note that the standard tetrahedron consists of points of the form $\sum_{i=1}^3 t_i v_i$, $0 \leq t_i \leq 1$. A neighborhood of a point on the 0-skeleton of a three dimensional simplicial complex can be modelled by a model space $\mathbf{B}$ where l copies of W are glued together along their faces according to the combinatorial information of the complex. Here, D is the point $(0,0,0)$.

Let $\mathbf{B}$ be a dimension-n, codimension-ν local model and let $\nu = n - k$. Recall that this means that $D \subset \mathbf{B}$ is of dimension k; more specifically, D can be isometrically identified with $\mathbf{R}^k$. We say $x \in \mathbf{B}$ is a codimension-$(n-j)$

singular point if $B_x(\sigma)$ is homeomorphic to $\mathbf{B}'(\sigma)$ where $\mathbf{B}'$ is some dimension-n, codimension-$(n-j)$ local model for some $\sigma > 0$. We denote the closure of the set of codimension-$(n-j)$ singular points by S_j and set $S_{-1} = \emptyset$. For example, if $\mathbf{B}$ is a codimension-$(n-k)$ local model, then $S_k = D$ and $S_i = \emptyset$ for $i = -1, ..., k-1$.

The following two definitions will be important in Section 4.

Definition 2 *Suppose $x \in S_{j+1} - S_j$. Thus, x is an interior point of a $(j+1)$-dimensional face F. We define $\pi_j(x)$ to be the set of all points x' in $S_j \cap F$ such that $|x - x'| = \min_{y \in S_j \cap F} |x - y|$. First, note that the closest point projection of x to the boundary of F is not necessarily unique so that $\pi_j(x)$ may contain more than one point. Secondly, because a face of a local model is a convex subset of Euclidean space, $\pi_j(x) \subset S_j - S_{j-1}$. For $i > j$, $x \in S_i$ and $x' \in S_j$, we write $x \rhd x'$ if we can arrive from x to x' by a sequence of successive projections, i.e. there exists a sequence*

$$x = y_i, y_{i-1}, ..., y_{j+1}, y_j = x'$$

so that

$$\begin{array}{rcl} y_i & \in & S_i, \\ y_{i-1} & \in & \pi_{i-1}(y_i) \subset S_{i-1}, \\ & \cdots & \\ y_{j+1} & \in & \pi_{j+1}(y_{i-j-1}) \subset S_{j+1}, \\ y_j & \in & \pi_j(y_{i-j}) \subset S_j. \end{array}$$

For $x \in \mathbf{B}(\sigma)$, let $\Pi_j(x)$ be the set of points $x' \in S_j$ so that $x \rhd x'$. For any set N, let $\Pi_j(N)$ be the set of points $x' \in S_j$ so that $x \rhd x'$ for $x \in N$.

Definition 3 *Let $x \in \mathbf{B}$ and $\sigma > 0$. The ball $B_x(\sigma)$ is called homogeneous if for all $t \in (0,1)$*

$$B_x(t\sigma) = tB_x(\sigma).$$

Given $x \in \mathbf{B}$, we let $St(x)$ denote the star of the point x in $\mathbf{B}$ ie. the union of the wedges containing x. Finally, if $x \in \mathbf{B}(r)$we set $R(x)$ to be the radius of the largest homogeneous ball centered at x contained in $St(x) \cap \mathbf{B}(r)$.

In addition to the Euclidean metric δ we equip a local model $\mathbf{B}$ (or $\mathbf{B}(r)$) with a Lipschitz Riemannian metric g. By this we mean that for each wedge W of $\mathbf{B}$ (resp. each face F of $\mathbf{B}$), we have a Lipschitz Riemannian metric g_W (resp. g_F) up to the boundary of W (resp. F) with the property that if F' is a face of W (resp. F) then the restriction g_W (resp. g_F) to F' is equal to $g_{F'}$. *Note that we do not necessarily assume that the faces of the wedges are totally geodesic.* We can express g as a matrix (g_{ij}) in terms of the Euclidean coordinate system on the wedges W inherited from the Euclidean space.

Definition 4 *We say $\lambda \in (0,1]$ is an ellipticity constant of g if for each wedge W (resp. each face F) the ellipticity constants of g_W (resp. g_F) are bounded below by λ and above by $\frac{1}{\lambda}$, in other words in terms of Euclidean coordinates on W (resp. F) we have*

$$\lambda^2|\xi|^2 \leq \sum_{i,j=1}^{n} g_{ij}\xi^i\xi^j \leq \frac{1}{\lambda^2}|\xi|^2. \tag{1}$$

Definition 5 *We say a metric g on $\mathbf{B}(r)$ is normalized if $g_{ij}(0) = \delta_{ij}$.*

2.2 Admissible Cell Complexes

A convex cell complex or simply a complex X in an affine space $\mathbf{E^d}$ is a finite collection $\{F^k\}$ of cells where each F^0 is a point, and each F^k is a bounded convex piecewise linear polyhedron with interior in some $\mathbf{E^k} \subset \mathbf{E^d}$, such that the boundary ∂F^k of F^k is a union of F^s with $s < k$ (called the faces of F^k), and such that if $s < k$ and $F^k \cap F^s \neq \emptyset$, then $F^s \subset F^k$. For example a simplicial complex is a cell complex whose cells are all simplices. We will denote by $X^{(i)}$ the i-dimensional skeleton of X, i.e. the union of all cells F^k where $k \leq i$. X is called n-dimensional or simply a n-complex if $X^{(n+1)} = \emptyset$ but $X^{(n)} \neq \emptyset$. The boundary of X, denoted ∂X, is the union of k-cells F^k, $k < n$, so that F is a face of exactly one n-cell. A point $p \in \partial X$ is called a boundary point and a point $p \in X - \partial X$ is called an interior point. In the sequel, we will require the following conditions (sometimes called admissibility conditions, or admissible complex (cf. [EF])):

(1) X is dimensionally homogeneous; i.e. for $k < n$, each k-cell is a face of a n-cell.

(2) X is locally $(n-1)$-chainable; i.e. for every connected, open set $U \subset X$, the open set $U - X^{(n-2)}$ is connected.

A Lipschitz Riemannian n-complex is a convex cell complex where each cell F is equipped with a Lipschitz Riemannian metric g_F up to the boundary. We are assuming that if F' is a face of F then the restriction g_F to F' is equal to $g_{F'}$. Admissible cell complexes are based on local models because of the following obvious proposition

Proposition 6 *Let X be an admissible Lipschitz Riemannian complex of dimension n with metric g given as (g_{ij}). Let $x \in X^{(k)} - X^{(k-1)}$ and let $\lambda \in (0,1]$ be the ellipticity constant of g near x. Then there exist a dimension-n, codimension-$(n-k)$ local model $\mathbf{B}$ and a homeomorphism $L_x : \mathbf{B}(\lambda R(x)) \to L_x(\mathbf{B}(\lambda R(x))) \subset X$ so that* *(i)* $L_x(0) = x$.

(ii) For any wedge W *of* $\mathbf{B}$, L_x *restricted to* $W \cap \mathbf{B}(\lambda R(x))$ *maps into the closure* $\overline{F}$ *of a* n*-cell of* X.

(iii) With W *viewed as a subset of* $\mathbf{R}^n$ *as in Section 2.1,* $L_x\big|_{W\cap\mathbf{B}(\lambda R(x))}$ *uniquely extends as an affine map* $\mathcal{L}_x$ *defined on* $\mathbf{R}^n$.

(iv) The pullback metric $h = L_x^* g$ *has the property that* $h_{ij}(0) = \delta_{ij}$ *with respect to the coordinate chart on* W.

Because g has ellipticity constant λ, $\mathcal{L}_x$ maps the ball of radius $\lambda R(x)$ centered at 0 into the largest ellipse contained in the ball of radius $R(x)$ centered at x.

We now mention that (a) trees and Bruhat-Tits buildings are examples of admissible cell complexes; (b) for any finitely generated group Γ there is a two dimensional admissible complex without boundary whose fundamental group is Γ (cf. [DM1]); (c) Triangulable Lipschitz manifolds and normal complex analytic spaces are homeomorphic to admissible complexes. For more details we refer to [EF].

2.3 Harmonic maps

We now define our target spaces.

Definition 7 *A complete metric space* (Y, d) *is said to be an NPC (non-positively curved) space if the following conditions are satisfied:*

(i) The space (Y, d) *is a length space. That is, for any two points* P *and* Q *in* Y*, there exists a rectifiable curve* γ_{PQ} *so that the length of* γ_{PQ} *is equal to* $d(P, Q)$ *(which we will sometimes denote by* d_{PQ} *for simplicity). We call such distance realizing curves geodesics.*

(ii) Let $P, Q, R \in Y$. *Define* Q_t *to be the point on the geodesic* γ_{QR} *satisfying* $d_{QQ_t} = td_{QR}$ *and* $d_{Q_tR} = (1-t)d_{QR}$. *Then*

$$d^2_{PQ_t} \le (1-t)d^2_{PQ} + td^2_{PR} - t(1-t)d^2_{QR}.$$

Remark. Simply connected Riemannian manifolds of non-positive sectional curvature, Bruhat-Tits Euclidean buildings associated with actions of p-adic Lie groups and $\mathbf{R}$-trees are examples of NPC spaces. These spaces are also referred to as CAT(0) spaces in literature. We refer to [BH] for more details.

We will now review the definition of harmonic maps. For details we refer the reader to [EF]. First, we define the energy of a map. Let (Y, d) be an NPC space and $f : (\mathbf{B}(r), g) \to Y$ be a L^2 map from the local model to Y. The energy ${}^gE^f$ is defined as the weak limit of the ϵ-approximate energy density measures which are measures derived from the appropriate average difference quotients. More specifically, define the ϵ-approximate energy $e_\epsilon : \mathbf{B}(r) \to \mathbf{R}$ by

$$e_\epsilon(x) = \begin{cases} \displaystyle\int_{y \in S(x,\epsilon)} \frac{d^2(f(x), f(y))}{\epsilon^2} \frac{d\sigma_{x,\epsilon}}{\epsilon^{n-1}} & \text{for } x \in \mathbf{B}(r)_\epsilon \\ 0 & \text{for } x \in \mathbf{B}(r) - \mathbf{B}(r)_\epsilon \end{cases}$$

where $\sigma_{x,\epsilon}$ is the induced measure on the ϵ-sphere $S(x, \epsilon)$ centered at x and $\mathbf{B}(r)_\epsilon = \{x \in \mathbf{B}(r) : d(x, \partial\mathbf{B}(r)) > \epsilon\}$. Define a family of functionals ${}^gE^f_\epsilon : C_c(\mathbf{B}(r)) \to \mathbf{R}$ by setting

$${}^gE^f_\epsilon(\varphi) = \int_{\mathbf{B}(r)} \varphi e_\epsilon d\mu_g.$$

Definition 8 *We say that $f : (\mathbf{B}(r), g) \to Y$ has finite energy (or that $f \in W^{1,2}(\mathbf{B}(r), Y)$ or simply that f is a $W^{1,2}$ map) if*

$${}^gE^f := \sup_{\varphi \in C_c(\mathbf{B}(r)), 0 \le \varphi \le 1} \limsup_{\epsilon \to 0} {}^gE^f_\epsilon(\varphi) < \infty.$$

Theorem 9 *Suppose $f : (\mathbf{B}(r), g) \to Y$ has finite energy. Then the measures $e_\epsilon(x)dx$ converge weakly to a measure which is absolutely continuous with respect to the Lebesgue measure. Therefore, there exists a function $e(x)$, which we call the energy density, so that $e_\epsilon(x)d\mu_g \rightharpoonup e(x)d\mu_g$.*

In analogy to the case of real valued functions, we write $|\nabla f|^2_g(x)$ in place of $e(x)$. (We will omit the subscript in $|\nabla f|^2_g$, $d\mu_g$ etc. if it is clear which metric we are using). In particular,

$${}^gE^f = \int_{\mathbf{B}(r)} |\nabla f|^2_g d\mu_g.$$

For a set $S \subset \mathbf{B}$, let

$${}^gE^f[S] = \int_S |\nabla f|^2_g d\mu_g.$$

We also define

$$|\nabla f|_g(x) = (|\nabla f|^2_g(x))^{1/2}.$$

For a Lipschitz vector field V on $\mathbf{B}(r)$, $|f_*(V)|^2_g$ is similarly defined. The real valued L^1 function $|f_*(V)|^2_g$ generalizes the norm squared on the directional derivative of f. We refer to [KS1] and [EF] for more details.

Theorem 10 *Suppose $f : (\mathbf{B}(r), g) \to Y$ has finite energy and V is a Lipschitz vector field. The operator ${}^g\pi^f$ defined by*

$${}^g\pi^f(V, W) = \frac{1}{2}|f_*(V + W)|^2_g - \frac{1}{2}|f_*(V - W)|^2_g.$$

is continuous, symmetric, bilinear, non-negative and tensorial. We call ${}^g\pi^f$ *the pull-back metric.*

Notation 11 *Let* $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ *be the standard Euclidean basis defined on each wedge inherited from* $\mathbf{R}^n$ *and* δ *the standard Euclidean metric. Set*

$$\frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^j} = {}^\delta\pi^f \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \quad \text{and} \quad \left|\frac{\partial f}{\partial x^i}\right|^2 = \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^i}.$$

Similarly for the standard Euclidean polar coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ *on each wedge we denote*

$$\frac{\partial f}{\partial x^k} \cdot \frac{\partial f}{\partial r} = {}^\delta\pi^f \left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial r}\right), \ \left|\frac{\partial f}{\partial r}\right|^2 = \frac{\partial f}{\partial r} \cdot \frac{\partial f}{\partial r} = {}^\delta\pi^f \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)$$

and

$$\frac{\partial f}{\partial \theta_i} \cdot \frac{\partial f}{\partial \theta_j} = {}^\delta\pi^f \left(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right).$$

Note that the energy density with respect to the metric g is given by

$$|\nabla f|_g^2 = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^j},$$

whereas the energy density with respect to the euclidean metric is given by

$$|\nabla f|^2 = |\nabla f|_\delta^2 = \sum_i \left|\frac{\partial f}{\partial x^i}\right|^2.$$

By using the identification with local models given in Proposition 6 all the above notions extend for any admissible complex X replacing $\mathbf{B}$ as a domain. We omit the details.

For the trace of $W^{1,2}$ maps we refer to the following theorem (cf. [KS1] and [EF]).

Theorem 12 *Let* Ω *be a compact domain in an admissible complex with Lipschitz Riemannian metric* g *and* (Y, d) *a metric space. Any* $f \in W^{1,2}(\Omega, Y)$ *has a well defined trace map denoted by* $Tr(f)$ *or simply* f*, with* $Tr(f) \in L^2(\partial\Omega, Y)$*. If the sequence* $f_i \in W^{1,2}(\Omega, Y)$ *has uniformly bounded energies* ${}^gE^{f_i}[\Omega]$ *and if* f_i *converges in* L^2 *to a map* f*, then* $Tr(f_i)$ *converges to* $Tr(f)$ *in* $L^2(\partial\Omega, Y)$*. Two maps* $f, g \in W^{1,2}(\Omega, Y)$ *have the same trace if and only if* $d(f, g) \in W^{1,2}(\Omega, \mathbf{R}) = W^{1,2}(\Omega)$ *has trace zero.*

We define $W_0^{1,2}(\Omega)$ to be the subset of $W^{1,2}(\Omega)$ functions with trace zero. The next two theorems are also contained in [KS1] and [EF].

Theorem 13 *Let Ω be a compact domain in an admissible complex with Lipschitz Riemannian metric g and (Y,d) a locally compact metric space. Let $f_i \in W^{1,2}(\Omega, Y)$ be a sequence satisfying $f_i \to f$ in L^2 and ${}^gE^{f_i}[\Omega] \leq C$ for some constant C independent of i. Then*

$$
{}^gE^f[\Omega] \leq \liminf_{i\to\infty} {}^gE^{f_i}[\Omega].
$$

Theorem 14 *Let Ω be a compact domain in anadmissible complex with Lipschitz Riemannian metric g and (Y,d) a locally compact metric space. Let $f_i \in W^{1,2}(\Omega, Y)$ be a sequence satisfying ${}^gE^{f_i}[\Omega] + \int_X d^2(f_i(x), Q)\ d\mu_g(x) \leq C$ for some fixed point Q of Y and some constant C independent of i. Then, there is a subsequence of f_i that converges in $L^2(X,Y)$ to a finite energy map f.*

Definition 15 *Let Ω be a compact domain in an admissible complex with Lipschitz Riemannian metric g and (Y,d) an NPC space. A map $f : \Omega \to Y$ is said to be harmonic if it is energy minimizing among all $W^{1,2}$-maps with the same trace (boundary condition).*

We end this section by proving two versions of the Poincaré inequality that we will need in the sequel.

Theorem 16 *Let Ω be a compact domain in an admissible complex with Lipschitz Riemannian metric g. Then, there is a constant C depending only on Ω and g so that for any $\varphi \in W_0^{1,2}(\Omega)$*

$$
\int_\Omega \varphi^2 d\mu_g \leq C \int_\Omega |\nabla\varphi|_g^2 d\mu_g.
$$

Proof. The proof follows closely the proof of the Poincare inequality in [Si] Lemma 2, therefore we will only give a sketch. Suppose the assertion is false: then for each $i = 1, 2...$, there exist functions $\varphi_i \in W_0^{1,2}(\Omega)$ so that

$$
\int_\Omega |\nabla\varphi_i|_g^2 d\mu_g < \frac{1}{i} \int_\Omega \varphi_i^2 d\mu_g.
$$

By setting

$$
v_i = \frac{\varphi_i}{(\int_\Omega \varphi_i^2 d\mu_g)^{1/2}}
$$

we have $\int_\Omega v_i^2 d\mu_g = 1$ and $\int_\Omega |\nabla v_i|_g^2 d\mu_g < \frac{1}{i}$. By Theorems 14 and 13 there exists a subsequence (which we denote again by i) so that $v_i \to v$ in $L^2(\Omega)$ and $\int_\Omega |\nabla v|_g^2 d\mu_g \leq \liminf_{i\to\infty} \int_\Omega |\nabla v_i|_g^2 d\mu_g = 0$. This implies that v must be constant, and since $\int_\Omega v^2 d\mu_g = \lim_{i\to\infty} \int_\Omega v_i^2 d\mu_g = 1$ it must be nonzero. On the other hand Theorem 12 implies that the trace of v is 0, which is a contradiction. Q.E.D.

Theorem 17 *Let X be a compact admissible complex with a Lipschitz Riemannian metric g and (Y,d) a metric space. Then, there is a constant C depending only on X and the ellipticity constant of the metric g so that for any $\varphi \in W^{1,2}(X,Y)$*

$$\inf_{P\in Y}\int_X d^2(\varphi,P)d\mu_g \leq C \int_X |\nabla\varphi|_g^2 d\mu_g.$$

PROOF. By [F], proof of Corollary 1 Step 2, the Poincare inequality holds for the Euclidean metric, i.e there exists a constant C depending only on X so that

$$\inf_{P\in Y}\int_X d^2(\varphi,P)d\mu \leq C\int_X |\nabla\varphi|^2 d\mu.$$

Let c be a universal constant depending only on the dimension of X. It follows from (1) that

$$\sum_{i,j=1}^n \delta_{ij}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_j} \leq c\lambda^{-2}\sum_{i,j=1}^n g^{ij}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_j}$$

and hence

$$|\nabla\varphi|^2 \leq c\lambda^{-2}|\nabla\varphi|_g^2. \tag{2}$$

Furthermore, (1) also implies that

$$\lambda^n \leq \frac{d\mu_g}{d\mu} \leq \lambda^{-n},$$

which combined with (2) completes the proof. Q.E.D.

Corollary 18 *Let $\mathbf{B}$ be a dimension-n, codimension-ν local model, g a Lipschitz Riemannian metric defined on $\mathbf{B}(r)$ and (Y,d) a metric space. Then, there is a constant C depending only on $\mathbf{B}(r)$ and the ellipticity constant of g so that for any $\varphi \in W^{1,2}(\mathbf{B}(r),Y)$ and σ sufficiently small*

$$\inf_{P\in Y}\int_{\mathbf{B}(\sigma)} d^2(\varphi,P)d\mu_g \leq C\sigma^2 \int_{\mathbf{B}(\sigma)} |\nabla\varphi|_g^2 d\mu_g.$$

Similarly, for any $\varphi \in W^{1,2}(\partial\mathbf{B}(r),Y)$ and σ sufficiently small

$$\inf_{P\in Y}\int_{\partial\mathbf{B}(\sigma)} d^2(\varphi,P)d\Sigma_g \leq C\sigma^2 \int_{\partial\mathbf{B}(\sigma)} |\nabla^\partial\varphi|_g^2 d\Sigma_g,$$

where ∇^∂ is the gradient tangential to $\partial\mathbf{B}(\sigma)$.

PROOF. Both inequalities follow immediately from Theorem 17 by rescaling and by the fact that $\mathbf{B}(\sigma)$ (resp. $\partial\mathbf{B}(\sigma)$) is piecewise smoothly diffeomorphic to the star (resp. the link) of the point 0. Q.E.D.

3 Monotonicity formula

In this section, we prove a monotonicity formula for harmonic maps. This is a modified version of the monotonicity formula shown in [GS] where the domain space is a Riemannian manifold. The technical difficulties posed by the singular nature of the domain space considered in this paper is that we cannot necessarily work in normal coordinates and that the faces are not necessarily totally geodesic in wedges with respect to the metric given.

Let $\mathbf{B}$ be a local model. We continue to use the Euclidean coordinates $(x^1, ..., x^n)$ in each wedge W. For $x, y \in \mathbf{B}$, we denote the induced (Euclidean) distance by $|x-y|$. By definition, if $x = (x^1, ..., x^n)$ and $y = (y^1, ..., y^n)$ are on the same wedge of $\mathbf{B}$, then $|x-y| = \sqrt{(x^1-y^1)^2 + ... + (x^n-y^n)^2}$. Furthermore, we let $(r, \theta_1, ..., \theta_{n-1})$ be the corresponding polar coordinates, i.e. r gives the radial distance from the origin and $\theta = (\theta_1, ..., \theta_{n-1})$ are the coordinates on the standard $(n-1)$-sphere. Let g be a normalized Lipschitz metric defined on $\mathbf{B}(r) = \{x \in \mathbf{B} : |x| < r\}$, i.e. if $g = (g_{ij})$ with respect to the coordinates $x = (x^1, ..., x^n)$ on a wedge W, then

$$|g_{ij}(x) - g_{ij}(\bar{x})| \leq c|x - \bar{x}|, \quad \forall x, \bar{x} \in W$$

and

$$|g_{ij}(x) - \delta_{ij}| \leq c\sigma \tag{3}$$

for $|x| \leq \sigma$. For $\sigma \in (0, r)$, we set

$${}^gE^f(\sigma) = \int_{\mathbf{B}(\sigma)} |\nabla f|_g^2 d\mu_g \tag{4}$$

and

$${}^gI^f(\sigma, Q) = \int_{\partial\mathbf{B}(\sigma)} d^2(f, Q) d\Sigma_g \tag{5}$$

for $Q \in Y$. Here $d\Sigma_g$ is the measure on $\partial\mathbf{B}(\sigma)$ induced by g. By [KS1] Lemma 2.5.1, there exists a unique point $Q_\sigma \in Y$ so that

$${}^gI^f(\sigma, Q_\sigma) = \inf_{Q \in Y} {}^gI^f(\sigma, Q). \tag{6}$$

Notation 19 *For the rest of this section we will use the notation*

$$E(\sigma) = {}^gE^f(\sigma) \quad \textit{and} \quad I(\sigma) = I(\sigma, Q) = {}^gI^f(\sigma, Q),$$

if Q is a generic point. Furthermore in all statements up to (including) Corollary 31, we will make the additional assumption that the metric g is normalized.

If we assume that the domain is a Riemannian manifold and replace $\mathbf{B}(\sigma)$ by a geodesic σ-ball, it is shown in [GS] that

$$\sigma \mapsto e^{C\sigma} \frac{\sigma E(\sigma)}{I(\sigma)} \tag{7}$$

is a non-decreasing function where C is some constant depending on the metric. Note that in our case $\mathbf{B}(\sigma)$ is a σ-ball with respect to the Euclidean metric δ on $\mathbf{B}$. The reason Euclidean balls are considered here is the possible incompatibility of the induced distance functions of the metrics given on two different wedges along a shared face. More specifically, let g_1 and g_2 be the metrics defined on wedges W_1 and W_2 sharing a face F. Since we do not assume that F is totally geodesic in W_1 or W_2, the induced distance functions in W_1 and W_2 do not necessarily agree in F.

We are thus considering a general Lipschitz metric g with no restriction on the faces and this leads to a modified version of the monotonicity formula which in turn gives a well defined version of the order (cf. Corollary 31). For a model space $\mathbf{B}$ with a Euclidean metric δ, the monotonicity of (7) follows from [M1].

We say a continuous function η defined on $\mathbf{B}(r)$ is smooth if the restriction of η to each wedge W of $\mathbf{B}(r)$ is smooth up to the boundary of W. The set of smooth functions with compact support in $\mathbf{B}(r)$ will be denoted by $C_c^\infty(\mathbf{B}(r))$.

Lemma 20 *Let $f : (\mathbf{B}(r), g) \to (Y, d)$ be a harmonic map. For any $\sigma \in (0, r)$ and $\eta \in C_c^\infty(\mathbf{B}(\sigma))$,*

$$\int_{\mathbf{B}(\sigma)} \left(|\nabla f|_g^2 (2-n)\eta - |\nabla f|_g^2 \sum_i x^i \frac{\partial \eta}{\partial x^i} + 2 \sum_{i,j,k} g^{ik} \frac{\partial \eta}{\partial x^i} x^j \frac{\partial f}{\partial x^j} \cdot \frac{\partial f}{\partial x^k} \right) d\mu_g + O(\sigma) E(\sigma) = 0 \quad (8)$$

where $|O(\sigma)| \leq c\sigma$ and c depends on $\mathbf{B}(r)$ and the Lipschitz bound of g.

PROOF. For t sufficiently small, we define $F_t : \mathbf{B}(r) \to \mathbf{B}(r)$ by setting

$$F_t(x) = (1 + t\eta(x))x$$

for each $x = (x^1, ..., x^n)$ in a wedge W. For $f_t : \mathbf{B}(r) \to Y$ defined as $f_t = f \circ F_t$, a direct computation (cf. [GS] Section 2) on each wedge W of $\mathbf{B}(r)$ gives

$$\begin{aligned} &\frac{d}{dt}\, {}^g E^{f_t}[W]|_{t=0} \\ &= \int_W \left(|\nabla f|_g^2 (2-n)\eta - |\nabla f|_g^2 \sum_i x^i \frac{\partial \eta}{\partial x^i} + 2 \sum_{i,j,k} g^{ik} \frac{\partial \eta}{\partial x^i} x^j \frac{\partial f}{\partial x^j} \cdot \frac{\partial f}{\partial x^k} \right) d\mu_g \\ &\quad + \text{remainder}. \end{aligned}$$

Here, the remainder term is given by

$$\int_W \left(-\eta \sum_{i,j,k} \frac{\partial g^{ij}}{\partial x^k} x^k \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^j} \sqrt{g} + |\nabla f|_g^2 \eta \sum_i x^i \frac{\partial \sqrt{g}}{\partial x^i} \right) dx.$$

Since we assume the metric g is Lipschitz, there exists a constant c so that $\left|\frac{\partial g^{ij}}{\partial x^k}\right|, \left|\frac{\partial \sqrt{g}}{\partial x^i}\right| \leq c$, which then implies that the remainder term is bounded by $c\sigma \times E(\sigma)$. Summing over all the wedges W of $\mathbf{B}(r)$ we get the right hand side of (8) and this equals to 0 since $f_0 = f$ is harmonic. Q.E.D.

Lemma 21 *If $f : (\mathbf{B}(r), g) \to (Y, d)$ satifies (8), then for $\sigma \in (0, r)$*

$$\left| \frac{E'(\sigma)}{E(\sigma)} - \frac{n-2}{\sigma} - \frac{2}{E(\sigma)} \int_{\partial \mathbf{B}(\sigma)} \sum_{i,k} g^{ik} \frac{x^i}{|x|} \frac{\partial f}{\partial x^k} \cdot \frac{\partial f}{\partial r} d\Sigma_g \right| \leq c_1 \tag{9}$$

for some constant c_1 depending on $\mathbf{B}(r)$ and the Lipschitz bound of g,

$$E'(\sigma) = (1 + O(\sigma)) \left(\frac{n-2+O(\sigma)}{\sigma} E(\sigma) + 2 \int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right) \tag{10}$$

and

$$(1 + c\sigma) \frac{E'(\sigma)}{E(\sigma)} \geq \frac{n-2}{\sigma} + \frac{2}{E(\sigma)} \int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g - c_1 \tag{11}$$

where c is as in (3).

Proof. Let η in (8) approximate the characteristic function of $\mathbf{B}(\sigma)$ to obtain

$$E'(\sigma) - \frac{n-2+O(\sigma)}{\sigma} E(\sigma) - 2 \int_{\partial \mathbf{B}(\sigma)} \sum_{i,j,k} g^{ik} \frac{x^i}{|x|} \frac{\partial f}{\partial x^k} \cdot \frac{\partial f}{\partial r} d\Sigma_g = 0 \tag{12}$$

which immediately implies inequality (9). Next, we use the inequality $g^{ik} \leq \delta^{ik} + c\sigma$ to show

$$\sum_{i,k} g^{ik} \frac{x^i}{|x|} \frac{\partial f}{\partial x^k} \cdot \frac{\partial f}{\partial r} \leq \left| \frac{\partial f}{\partial r} \right|^2 + c\sigma \sum_k \left| \frac{\partial f}{\partial x^k} \cdot \frac{\partial f}{\partial r} \right| \leq \left| \frac{\partial f}{\partial r} \right|^2 + c\sigma \left| \nabla f \right|^2.$$

Using this, inequalities (10) and (11) follow again from (12). Q.E.D.

Lemma 22 *Let $f : (\mathbf{B}(r), g) \to (Y, d)$ be a harmonic map. For any $Q \in Y$, $\triangle d^2(f, Q) - 2|\nabla f|_g^2 \geq 0$ weakly, i.e.*

$$2 \int_{\mathbf{B}(r)} |\nabla f|_g^2 \eta \; d\mu_g \leq - \int_{\mathbf{B}(r)} < \nabla d^2(f, Q), \nabla \eta >_g \; d\mu_g \tag{13}$$

for any $\eta \in C_c^\infty(\mathbf{B}(r))$.

PROOF. This inequality follows from a target variation of the harmonic map and hence the singular nature of the domain is not essential in the proof. Details can be found in the proof of [GS] Proposition 2.2. Q.E.D.

Lemma 23 *If $f : (\mathbf{B}(r), g) \to (Y, d)$ satisfies (13), then*

$$E(\sigma) \le I(\sigma)^{\frac{1}{2}} \left(\left(\int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\sigma_g \right)^{\frac{1}{2}} + c\sigma (E'(\sigma))^{\frac{1}{2}} \right) \tag{14}$$

and

$$2E(\sigma) \le \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q)) d\Sigma_g + I(\sigma) + k\sigma^2 E'(\sigma) \tag{15}$$

for some constants c, k depending on $\mathbf{B}(r)$ and the Lipschitz bound of g and

$$\frac{1}{I(\sigma)} \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g \le \frac{2}{E(\sigma)} \int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\mu_g + 2c\sigma \frac{E'(\sigma)}{E(\sigma)} \tag{16}$$

where c is as in (3).

PROOF. Let η in (13) approximate the characteristic function of $\mathbf{B}(\sigma)$ to obtain

$$2E(\sigma) \le \int_{\partial \mathbf{B}(\sigma)} < \nabla d^2(f, Q), \nabla |x| >_g d\Sigma_g = \int_{\partial \mathbf{B}(\sigma)} g^{ij} \frac{\partial}{\partial x^i} d^2(f, Q) \frac{x^j}{|x|} d\Sigma_g.$$

Using the estimate $g^{ij} \le \delta^{ij} + c\sigma$, we obtain

$$\begin{aligned} 2E(\sigma) &\le \int_{\partial \mathbf{B}(r)} \frac{\partial}{\partial r} d^2(f, Q)) d\Sigma_g + c\sigma \int_{\partial \mathbf{B}(r)} \sum_i \left| \frac{\partial}{\partial x^i} d^2(f, Q)) \right| d\Sigma_g \quad (17) \\ &\le 2 \int_{\partial \mathbf{B}(r)} d(f, Q) \frac{\partial}{\partial r} d(f, Q)) d\Sigma_g \\ &\quad + 2c\sigma \int_{\partial \mathbf{B}(\sigma)} d(f, Q) \sum_i \left| \frac{\partial}{\partial x^i} d(f, Q) \right| d\Sigma_g \\ &\le 2I(\sigma)^{\frac{1}{2}} \left(\int_{\partial \mathbf{B}(r)} \left| \frac{\partial}{\partial r} d(f, Q)) \right|^2 d\Sigma_g \right)^{\frac{1}{2}} \\ &\quad + 2c\sigma I(\sigma)^{\frac{1}{2}} \left(\int_{\partial \mathbf{B}(\sigma)} \sum_i \left| \frac{\partial}{\partial x^i} d(f, Q) \right|^2 d\Sigma_g \right)^{\frac{1}{2}}. \end{aligned}$$

The triangle inequality implies that

$$\left| \frac{\partial}{\partial x^i} d(f, Q) \right|^2 \le \left| \frac{\partial f}{\partial x^i} \right|^2 \quad \text{and} \quad \left| \frac{\partial}{\partial r} d(f, Q) \right|^2 \le \left| \frac{\partial f}{\partial r} \right|^2.$$

From this, (14) follows immediately. Additionally, use the Cauchy-Schwarz inequality to obtain

$$2c\sigma d(f,Q)\left|\frac{\partial}{\partial x^i}d(f,Q)\right| \leq d^2(f,Q)+c^2\sigma^2\left|\frac{\partial f}{\partial x^i}\right|^2$$

which implies

$$2E(\sigma) \leq \int_{\mathbf{B}(r)} \frac{\partial}{\partial r}d^2(f,Q))d\Sigma_g + \int_{\partial\mathbf{B}(\sigma)} d^2(f,Q)d\Sigma_g + c^2\sigma^2 \int_{\partial\mathbf{B}(\sigma)} |\nabla f|^2 d\Sigma_g.$$

From this, (15) follows immediately. Lastly, again use (17) to obtain

$$\begin{aligned}
&E(\sigma)\int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}d^2(f,Q)d\Sigma_g \\
\leq\ & 2\left(\int_{\partial\mathbf{B}(\sigma)} d(f,Q)\frac{\partial}{\partial r}d(f,Q)d\Sigma_g\right)^2 \\
&+2c\sigma\left(\int_{\partial\mathbf{B}(\sigma)} d(f,Q)\frac{\partial}{\partial r}d(f,Q)d\Sigma_g\right)\left(\int_{\partial\mathbf{B}(\sigma)} d(f,Q)\sum_i\left|\frac{\partial}{\partial x^i}d(f,Q)\right|d\Sigma_g\right) \\
\leq\ & 2I(\sigma)\int_{\partial\mathbf{B}(\sigma)}\left|\frac{\partial f}{\partial r}\right|^2 d\Sigma_g + 2c\sigma I(\sigma)\int_{\partial\mathbf{B}(\sigma)} |\nabla f|^2 d\Sigma_g.
\end{aligned}$$

This immediately implies (16). Q.E.D.

The following energy growth estimate is also given in [Ch] with geodesic balls (and not Euclidean balls as it is here).

Lemma 24 *Let $f : (\mathbf{B}(r), g) \to (Y, d)$ be a harmonic map. There exist $\sigma_0 > 0$ and $\gamma > 0$ depending on $\mathbf{B}(r)$, the Lipschitz bound and the ellipticity constant of g so that*

$$\sigma \mapsto \frac{E(\sigma)}{\sigma^{n-2+2\gamma}},\ \ \sigma \in (0, \sigma_0)$$

is non-decreasing.

PROOF. Let $Q_\sigma \in Y$ so that

$$I(\sigma, Q_\sigma) = \inf_{Q \in Y} I(\sigma, Q).$$

Thus, the Poincaré inequality (cf. Corollary 18) implies that there exists $C_0 > 0$ so that

$$I(\sigma, Q_\sigma) \leq C_0\sigma^2 \int_{\partial\mathbf{B}(\sigma)} |\nabla^\partial f|_g^2 d\Sigma_g$$

where $|\nabla^{\partial} f|_g^2$ is the tangential part of $|\nabla f|_g^2$. If we write

$$h_{ij} = g(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j})$$

then

$$|\nabla^{\partial} f|_g^2 = \frac{1}{r^2} h^{ij} \frac{\partial f}{\partial\theta_i} \cdot \frac{\partial f}{\partial\theta_j} = (1 + O(\sigma)) \left(|\nabla f|^2 - \left| \frac{\partial f}{\partial r} \right|^2 \right).$$

Thus,

$$I(\sigma, Q_\sigma) \leq C_0 \sigma^2 (1 + O(\sigma)) \int_{\partial \mathbf{B}(\sigma)} \left(|\nabla f|^2 - \left| \frac{\partial f}{\partial r} \right|^2 \right) d\Sigma_g. \tag{18}$$

Therefore,

$$\begin{aligned}
E^2(\sigma) &\leq I(\sigma, Q_\sigma) \left(\left(\int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right)^{\frac{1}{2}} + c\sigma E'(\sigma)^{\frac{1}{2}} \right)^2 \quad \text{by (14)} \\
&\leq 2I(\sigma, Q_\sigma) \left(\int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g + c^2 \sigma^2 E'(\sigma) \right) \\
&\leq C\sigma^2 \left((1 + O(\sigma)) \left(E'(\sigma) - \int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right) \right) \quad \text{by (18)} \\
&\quad \times \left(\int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g + c^2 \sigma^2 E'(\sigma) \right) \\
&\leq C\sigma \left((n - 2 + O(\sigma)) E(\sigma) + \sigma (1 + O(\sigma)) \int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right) \quad \text{by (10)} \\
&\quad \times \left((1 + O(\sigma)) \int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g + c^2 \sigma (n - 2 + O(\sigma)) E(\sigma) \right) \\
&\leq C' \left(\sigma^2 E^2(\sigma) + \sigma E(\sigma) \int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g + \sigma^2 \left(\int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right)^2 \right) \\
&\leq C' \left((\sigma^2 + \epsilon) E^2(\sigma) + \left(\frac{4}{\epsilon} + 1 \right) \sigma^2 \left(\int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right)^2 \right).
\end{aligned}$$

Note that the constant C and C' depend only on the Lipschitz constant of g and the constant coming from the Poincaré inequality which only depends on the ellipticity constant of g and the number of wedges of $\mathbf{B}(r)$. Thus, the constants below also depend only on these quantities. By choosing $\sigma > 0$ sufficiently small (depending on C'), we see that there exists a constant K so that

$$E(\sigma) \leq K\sigma \int_{\partial \mathbf{B}(\sigma)} \left|\frac{\partial f}{\partial r}\right|^2 d\Sigma_g.$$

Using (10), we also have

$$\begin{aligned}
\sigma E'(\sigma) &= (n-2+O(\sigma))E(\sigma) + (2\sigma + O(\sigma^2)) \int_{\partial \mathbf{B}(\sigma)} \left|\frac{\partial f}{\partial r}\right|^2 d\Sigma_g \\
&\geq (n-2+O(\sigma))E(\sigma) + \frac{2+O(\sigma)}{K} E(\sigma) \\
&= (n-2+\frac{2}{K}+O(\sigma))E(\sigma) \\
&\geq (n-2+2\gamma)E(\sigma)
\end{aligned}$$

for $\gamma, \sigma > 0$ sufficiently small. This implies

$$\frac{d}{d\sigma}\left(\log \frac{E(\sigma)}{\sigma^{n-2+2\gamma}}\right) \geq 0$$

for $\sigma > 0$ sufficiently small. Q.E.D.

Lemma 25 *For sufficiently small $\sigma > 0$ depending on the Lipschitz bound of g and $\mathbf{B}$ and for any map $f : \mathbf{B}(r) \to (Y, d)$, we have*

$$\left| \frac{I'(\sigma)}{I(\sigma)} - \frac{n-1}{\sigma} - \frac{1}{I(\sigma)} \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g \right| \leq c_2 \tag{19}$$

for some constant c_2 depending on $\mathbf{B}(r)$ and the Lipschitz bound of g.

Proof. Let $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_{n-1}}\}$ be the tangent basis corresponding to the polar coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ on W. We define

$$v(r,\theta) = \frac{1}{r^{n-1}} \sqrt{\left| \det \left(g(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}) \right) \right|}.$$

By the fact that $g_{ij}(0) = \delta_{ij}$, we have

$$|v(\sigma,\theta) - 1| = |v(\sigma,\theta) - \lim_{r \to 0} v(r,\theta)| \leq c'\sigma. \tag{20}$$

for some constant c' depending on the Lipschitz bound of g. Since the measure induced on $\partial \mathbf{B}(\sigma)$ by g can be written $d\Sigma_g = \sigma^{n-1} v(\sigma,\theta) d\theta$ where $d\theta$ is volume form on the standard $(n-1)$-sphere, we have

$$\frac{\partial}{\partial \sigma} d\Sigma_g = (n-1)\sigma^{n-2} v(\sigma,\theta) d\theta + \sigma^{(n-1)} \frac{\partial v}{\partial \sigma}(\sigma,\theta) d\theta$$

and

$$d\Sigma_g \geq \sigma^{n-1}(1 - c'\sigma)d\theta \geq \frac{1}{2}\sigma^{n-1}d\theta$$

for sufficiently small $\sigma > 0$. Thus,

$$\begin{aligned}
&\frac{d}{d\sigma}\int_{\partial\mathbf{B}(\sigma)} d^2(f,Q)d\Sigma_g \\
&= \int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}(d^2(f,Q))d\Sigma_g + \frac{n-1}{\sigma}\int_{\partial\mathbf{B}(\sigma)} d^2(f,Q)d\Sigma_g \\
&\quad + \int_{\partial\mathbf{B}(1)} d^2(f,Q)\sigma^{n-1}\frac{\partial v}{\partial r}(\sigma,\theta)d\theta
\end{aligned}$$

which in turn implies

$$\begin{aligned}
&\left|\frac{I'(\sigma)}{I(\sigma)} - \frac{n-1}{\sigma} - \frac{1}{I(\sigma)}\int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}(d^2(f,Q))d\Sigma_g\right| \\
&= \frac{1}{I(\sigma)}\left|\int_{\partial\mathbf{B}(1)} d^2(f,Q)\sigma^{n-1}\frac{\partial v}{\partial r}(\sigma,\theta)d\theta\right| \\
&\leq \frac{c}{I(\sigma)}\int_{\partial\mathbf{B}(1)} d^2(f,Q)\sigma^{n-1}d\theta \\
&\leq \frac{2c}{I(\sigma)}\int_{\partial\mathbf{B}(\sigma)} d^2(f,Q)d\Sigma_g = 2c
\end{aligned}$$

for sufficiently small σ. This immediately implies (19). Q.E.D.

Let $f : (\mathbf{B}(r), g) \to (Y, d)$ be a harmonic map. Inequality (19) implies that there exists σ_0 sufficiently small so that for $\sigma < \sigma_0$,

$$\frac{I'(\sigma)}{I(\sigma)} \leq \frac{n-1}{\sigma} + \frac{1}{I(\sigma)}\int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}d^2(f,Q)d\Sigma_g + c_2. \tag{21}$$

Together, (11), (16) and (21) imply

$$(1 + 3c\sigma)\frac{E'(\sigma)}{E(\sigma)} - \frac{I'(\sigma)}{I(\sigma)} + \frac{1}{\sigma} + c_3 \geq 0 \tag{22}$$

where $c_3 = c_1 + c_2$. We use this inequality to prove a modified monotonicity which we describe below. For notational simplicity by rescaling the metric g, we may assume that $3c \leq 1$ and $\sigma_0 = 1$. Let

$$J(\sigma) = \max_{s\in[0,\sigma]} I(s)$$

and set

$$A = \left\{\sigma : \frac{E'(\sigma)}{E(\sigma)} - \frac{J'(\sigma)}{J(\sigma)} + \frac{1}{\sigma} + c_3 \leq 0\right\}.$$

Roughly speaking, A is the *bad set* where the ordinary monotonicity formula fails. Notice that if $J'(s) \neq 0$ then $I(s)$ is increasing and hence $I(s) = J(s)$, $I'(s) = J'(s)$. If $J'(s) = 0$, then $I'(s) \leq 0$ (for, if $I'(s) > 0$ then J would be strictly increasing near s). Therefore we obtain the following pair of inequalities:

$$\frac{E'(\sigma)}{E(\sigma)} - \frac{I'(\sigma)}{I(\sigma)} + \frac{1}{\sigma} + c_3 \geq 0 \qquad \text{for } \sigma \notin A \tag{23}$$

$$(1+\sigma)\frac{E'(\sigma)}{E(\sigma)} - \frac{I'(\sigma)}{I(\sigma)} + \frac{1}{\sigma} + c_3 \geq 0 \qquad \text{for } \sigma \in A. \tag{24}$$

For $\sigma \in (0,1)$, set

$$F(\sigma) = E(\sigma) \exp\left(- \int_{A\cap(\sigma,1)} s \frac{E'(s)}{E(s)} ds \right).$$

Lemma 26 *For $F(\sigma)$ defined above,*

$$\frac{F'(\sigma)}{F(\sigma)} = \begin{cases} \frac{E'(\sigma)}{E(\sigma)} & \text{for } \sigma \notin A \\ (1+\sigma)\frac{E'(\sigma)}{E(\sigma)} & \text{for } \sigma \in A. \end{cases} \tag{25}$$

Consequently,

$$\sigma \mapsto e^{c_3\sigma} \frac{\sigma F(\sigma)}{I(\sigma)}$$

is non-decreasing for σ sufficiently small. Furthermore, for Q_σ as in (6)

$$\sigma \mapsto e^{c_3\sigma} \frac{\sigma F(\sigma)}{I(\sigma, Q_\sigma)}$$

is also non-decreasing for σ sufficiently small.

Remark. If g is the Euclidean metric δ, then $c = 0$ above and

$$\sigma \mapsto \frac{\sigma E(\sigma)}{I(\sigma)}$$

is non-decreasing.

PROOF. We first note that as it is an integral of an L^1 function $\sigma \mapsto E(\sigma)$ is absolutely continuous on $[0, r]$. Let

$$\varphi(\sigma) = -\int_{A\cap(\sigma,1)} s\frac{E'(s)}{E(s)} ds.$$

Then, for $a \leq \sigma < \sigma' \leq b$, we have

$$\begin{aligned} &|\varphi(\sigma) - \varphi(\sigma')| \\ &\leq \left| \int_{A\cap(\sigma,\sigma')} s\frac{E'(s)}{E(s)} ds \right| \leq \frac{b}{E(a)} \left| \int_\sigma^{\sigma'} E'(s) ds \right| = \frac{b}{E(a)} |E(\sigma') - E(\sigma)|, \end{aligned}$$

which implies that the function $\varphi(\sigma)$ and hence also $F(\sigma)$ is absolutely continuous. Here we are assuming that f is non-constant hence $E(a) \neq 0$. Thus,

$$\varphi'(\sigma) = \lim_{\epsilon \to 0} \frac{\varphi(\sigma+\epsilon) - \varphi(\sigma-\epsilon)}{2\epsilon} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{A \cap [\sigma-\epsilon, \sigma+\epsilon]} -s\frac{E'(s)}{E(s)} ds.$$

Therefore $\varphi'(\sigma) = 0$ for a.e. $\sigma \notin A$ and $\varphi'(\sigma) = -\sigma \frac{E'(\sigma)}{E(\sigma)}$ for a.e. $\sigma \in A$. This implies (25). Finally, note that the function $\sigma \mapsto \log\left(e^{c_3\sigma}\frac{\sigma F(\sigma)}{I(\sigma)}\right)$ is absolutely continuous on any interval $[a, b] \subset (0, r)$. Hence by combining (25), (23) and (24), we obtain

$$\frac{d}{ds} \log \frac{\sigma F(\sigma)}{I(\sigma)} \geq \frac{F'(\sigma)}{F(\sigma)} - \frac{I'(\sigma)}{I(\sigma)} + \frac{1}{\sigma} \geq -c_3,$$

which implies the monotonicity of $\sigma \mapsto e^{c_3\sigma}\frac{\sigma F(\sigma)}{I(\sigma)}$. Furthermore, since $I(\sigma_2, Q_{\sigma_1}) \geq I(\sigma_2, Q_{\sigma_2})$, we have for $\sigma_1 < \sigma_2$,

$$e^{c_3\sigma_1}\frac{\sigma_1 F(\sigma_1)}{I(\sigma_1, Q_{\sigma_1})} \leq e^{c_3\sigma_2}\frac{\sigma_2 F(\sigma_2)}{I(\sigma_2, Q_{\sigma_1})} \leq e^{c_3\sigma_2}\frac{\sigma_2 F(\sigma_2)}{I(\sigma_2, Q_{\sigma_2})}.$$

This implies the monotonicity of $\sigma \mapsto e^{c_3\sigma}\frac{\sigma F(\sigma)}{I(\sigma, Q_\sigma)}$. Q.E.D.

Lemma 26 shows the monotonicity involving a corrected energy term $F(\sigma)$. We now want to show that the correction factor is well-behaved as $\sigma \to 0$.

Lemma 27

$$\int_{A\cap(0,1)} s\frac{E'(s)}{E(s)} ds := \lim_{\sigma \to 0^+} \int_{A\cap(\sigma,1)} s\frac{E'(s)}{E(s)} ds < \infty.$$

Proof. For $s \in A$,

$$s\frac{E'(s)}{E(s)} \leq s\frac{J'(s)}{J(s)} - 1 - c_3 s \leq s\frac{J'(s)}{J(s)}. \tag{26}$$

by the definition of the set A. Thus, it is sufficient to prove

$$\lim_{\sigma \to 0^+} \int_{A\cap(\sigma,1)} s\frac{J'(s)}{J(s)} ds < \infty. \tag{27}$$

We follow the argument of Proposition 3.1 in [M2]. Let M be sufficiently large so that for $\sigma \in (0, 1]$,

$$\int_{\mathbf{B}(\sigma)} |\nabla f|^2 d\mu_g \leq ME(\sigma),$$

$$N := e^{c_3}\frac{F(1)}{J(1)} \geq e^{c_3\sigma}\frac{\sigma F(\sigma)}{J(\sigma)} \geq \frac{\sigma F(\sigma)}{J(\sigma)}$$

and $K = MN$. Furthermore, let $0 < \theta_1 < \theta_2 \leq 1$ and $r_0 \in (\theta_1, \theta_2]$. For $s \in (\theta_1, r_0)$, we have by (19) that

$$\begin{aligned} I'(s) &\leq \int_{\partial \mathbf{B}(s)} 2d(f,P)\frac{\partial}{\partial r}d(f,P)d\Sigma_g + \frac{n-1+c_2 s}{s}I(s) \\ &\leq \int_{\partial \mathbf{B}(s)} \left(\frac{1}{\epsilon}d^2(f,P) + \epsilon\left(\frac{\partial}{\partial r}d(f,P)\right)^2\right)d\Sigma_g + \frac{n-1+c_2 s}{\theta_1}I(s) \\ &\leq \int_{\partial \mathbf{B}(s)} \left(\frac{1}{\epsilon}d^2(f,P) + \epsilon|\nabla f|^2\right)d\Sigma_g + \frac{n-1+c_2 s}{\theta_1}I(s) \\ &\leq \epsilon\int_{\partial \mathbf{B}(s)} |\nabla f|^2 d\Sigma_g + \left(\frac{1}{\epsilon} + \frac{C}{\theta_1}\right)I(s) \end{aligned}$$

for some sufficiently large C. Therefore,

$$I(r_0) - I(\theta_1) = \int_{\theta_1}^{r_0} I'(s)ds \leq \epsilon M E(r_0) + \left(\frac{1}{\epsilon} + \frac{C}{\theta_1}\right)\int_{\theta_1}^{r_0} I(s)ds. \tag{28}$$

Hence

$$I(r_0) - \epsilon M E(r_0) \leq I(\theta_1) + \left(\frac{1}{\epsilon} + \frac{C}{\theta_1}\right)(r_0 - \theta_1)\max_{s\in[\theta_1, r_0]} I(s).$$

Since $r_0 \in (\theta_1, \theta_2]$ is arbitrary,

$$\max_{s\in[\theta_1,\theta_2]} I(s) - \epsilon M E(\theta_2) \leq I(\theta_1) + \left(\frac{1}{\epsilon} + \frac{C}{\theta_1}\right)(\theta_2 - \theta_1)\max_{s\in[\theta_1,\theta_2]} I(s),$$

which then implies

$$\left[1 - \left(\frac{1}{\epsilon} + \frac{C}{\theta_1}\right)(\theta_2 - \theta_1)\right]\max_{s\in[\theta_1,\theta_2]} I(s) - \epsilon M E(\theta_2) \leq I(\theta_1).$$

If $\max_{s\in[0,\theta_1]} I(s) \geq \max_{s\in[\theta_1,\theta_2]} I(s)$ then $J(\sigma)$ is identically equal to a constant in $[\theta_1, \theta_2]$. If $\max_{s\in[0,\theta_1]} I(s) \leq \max_{s\in[\theta_1,\theta_2]} I(s)$, then $\max_{s\in[\theta_1,\theta_2]} J(s) = \max_{s\in[\theta_1,\theta_2]} I(s)$. Either way, we have

$$\left[1 - \left(\frac{1}{\epsilon} + \frac{C}{\theta_1}\right)(\theta_2 - \theta_1)\right]\max_{s\in[\theta_1,\theta_2]} J(s) - \epsilon M E(\theta_2) \leq J(\theta_1),$$

which immediately implies

$$\left[1 - \left(\frac{1}{\epsilon} + \frac{C}{\theta_1}\right)(\theta_2 - \theta_1)\right]J(\theta_2) - \epsilon M E(\theta_2) \leq J(\theta_1).$$

For $\theta_0 \in (0,1)$ to be determined later, we set

$$\epsilon = \frac{\theta_0^{\frac{j(1-\theta_0^n)}{1-\theta_0}+n}}{2K}$$

to obtain

$$\left[1-\left(2K\theta_0^{\frac{-j(1-\theta_0^n)}{1-\theta_0}-n}+\frac{C}{\theta_1}\right)(\theta_2-\theta_1)\right]J(\theta_2)-\frac{E(\theta_2)}{2N}\theta_0^{\frac{j(1-\theta_0^n)}{1-\theta_0}+n}\leq J(\theta_1) \quad (29)$$

Let

$$\phi(\theta,n,j)=\frac{1}{2}-\left(2K\theta^{\frac{-j(1-\theta^n)}{1-\theta}}+\frac{C}{\theta}\right)(1-\theta).$$

Then

$$\lim_{\theta\to 1}\phi(\theta,n,j)=\frac{1}{2}$$

uniformly independently of j,n. Therefore, there exists θ_0 sufficiently close to 1 so that $\phi(\theta_0,n,j)>\frac{1}{4}$ independently of n and j. Choose j so that $\theta_0^j<\frac{1}{4}$. Then

$$\frac{1}{2}-\left(2K\theta_0^{\frac{-j(1-\theta_0^{2n})}{1-\theta_0}}+\frac{C}{\theta_0}\right)(1-\theta_0)>\theta_0^j \quad (30)$$

for any n.

Since $F(1)=E(1)$ by definition, we have that

$$\frac{E(1)}{N}=\frac{J(1)}{e^{c_3}}\leq J(1).$$

Thus, (29) with $n=0$, $\theta_1=\theta_0$ and $\theta_2=1$ implies

$$\left[\frac{1}{2}-\left(2K\theta_0^{\frac{-j}{1-\theta_0}}+\frac{C}{\theta_0}\right)(1-\theta_0)\right]J(1)\leq J(\theta_0)$$

and by inequality (30), $\theta_0^jJ(1)<J(\theta_0)$. Now suppose $\theta_0^jJ(\theta_0^k)<J(\theta_0^{k+1})$ for $k=0,...,n-1$. Then

$$\begin{aligned}
\int_{\theta_0^n}^1 s\frac{d}{ds}\log J(s)ds &= \sum_{k=0}^{n-1}\int_{\theta_0^{k+1}}^{\theta_0^k} s\frac{d}{ds}\log J(s)ds\\
&\leq \sum_{k=0}^{n-1}\theta_0^k\int_{\theta_0^{k+1}}^{\theta_0^k}\frac{d}{ds}\log J(s)ds\\
&\leq \sum_{k=0}^{n-1}\theta_0^k\log\frac{J(\theta_0^k)}{J(\theta_0^{k+1})}\\
&\leq \sum_{k=0}^{n-1}\theta_0^k\log\theta_0^{-j}\quad \text{(by the inductive hypothesis)}\\
&= \sum_{k=0}^{n-1}\log\theta_0^{-j\theta_0^k}
\end{aligned}$$

$$
\begin{aligned}
&= \log \theta_0^{-j\sum_{k=0}^{n-1}\theta_0^k} \\
&= \log \theta_0^{\frac{-j(1-\theta_0^n)}{1-\theta_0}}
\end{aligned}
$$

Using the fact that $\frac{J'(s)}{J(s)} \geq 0$, we obtain

$$
\begin{aligned}
\log \frac{F(1)}{F(\theta_0^n)} &= \log \frac{E(1)}{E(\theta_0^n)} + \int_{A\cap(\theta_0^n,1)} s\frac{E'(s)}{E(s)}ds \\
&\leq \log \frac{E(1)}{E(\theta_0^n)} + \int_{A\cap(\theta_0^n,1)} s\frac{J'(s)}{J(s)}ds \quad \text{(by (26))} \\
&\leq \log \frac{E(1)}{E(\theta_0^n)} + \int_{\theta_0^n}^{1} s\frac{J'(s)}{J(s)}ds \\
&\leq \log \frac{E(1)}{E(\theta_0^n)} + \log \theta_0^{\frac{-j(1-\theta_0^n)}{1-\theta_0}}.
\end{aligned}
$$

Therefore, using the fact that $E(1) = F(1)$ and the definition of N, we obtain

$$\theta_0^{\frac{j(1-\theta_0^n)}{1-\theta_0}} E(\theta_0^n) \leq F(\theta_0^n) \leq \frac{NJ(\theta_0^n)}{\theta_0^n}.$$

Thus, we can use the inequality

$$\frac{E(\theta_0^n)}{2N}\theta_0^{\frac{j(1-\theta_0^n)}{1-\theta_0}+n} \leq \frac{J(\theta_0^n)}{2}$$

in (29) with $\theta_1 = \theta_0^{n+1}$ and $\theta_2 = \theta_0^n$ to obtain

$$\left[\frac{1}{2} - \left(2K\theta_0^{\frac{-j(1-\theta_0^{2n})}{1-\theta_0}} + \frac{C}{\theta_0}\right)(1-\theta_0).\right] J(\theta_0^n) \leq J(\theta_0^{n+1})$$

Hence, by inequality (30), we have

$$\theta_0^j J(\theta_0^n) < J(\theta_0^{n+1}). \tag{31}$$

By induction, inequality (31) holds for all n which in turn inplies that

$$\int_{\theta_0^n}^1 s\frac{d}{ds}\log J(s)ds \leq \log \theta_0^{\frac{-j(1-\theta_0^n)}{1-\theta_0}}$$

holds for all n. Letting $n \to \infty$, we obtain

$$\int_{A\cap(0,1)} s\frac{d}{ds}\log J(s)ds \leq \int_0^1 s\frac{d}{ds}\log J(s)ds \leq \log \theta_0^{\frac{-j}{1-\theta_0}} < \infty.$$

This proves (27) and the proof is complete. Q.E.D.

Corollary 28

$$\alpha := \lim_{\sigma \to 0} \frac{\sigma E(\sigma)}{I(\sigma, Q_\sigma)} < \infty$$

exists.

PROOF. By Lemmas 26 and 27

$$\begin{aligned}
\lim_{\sigma \to 0} \frac{\sigma E(\sigma)}{I(\sigma, Q_\sigma)} &= \lim_{\sigma \to 0} \left(\frac{\sigma F(\sigma)}{I(\sigma, Q_\sigma)} \cdot \frac{E(\sigma)}{F(\sigma)} \right) \\
&= \lim_{\sigma \to 0} e^{c_3 \sigma} \frac{\sigma F(\sigma)}{I(\sigma, Q_\sigma)} \cdot \lim_{\sigma \to 0} \exp \left(\int_{A \cap (\sigma, 1)} s \frac{E'(s)}{E(s)} ds \right) < \infty.
\end{aligned}$$

Q.E.D.

Definition 29 *We call α of Corollary 28 the order of f at 0 denoted by $\alpha = ord^f(0)$.*

Lemma 30 *Let $\alpha = ord^f(0)$. There exist constants c_0 and σ_0 depending only on $\mathbf{B}(r)$, the Lipschitz bound and the ellipticity constant of g so that if*

$$\tilde{E}(\sigma) := E(\sigma) \exp \left(c_0 \int_{A \cap (0, \sigma)} s \frac{E'(s)}{E(s)} ds \right),$$

then

$$\sigma \mapsto e^{c_0 \sigma} \frac{\tilde{E}(\sigma)}{\sigma^{2\alpha + n - 2}}$$

is non-decreasing for $\sigma \in (0, \sigma_0)$.

PROOF. Set

$$G(\sigma) = E(\sigma) \exp \left(\int_{A \cap (0, \sigma)} s \frac{E'(s)}{E(s)} ds \right).$$

Since

$$G(\sigma) = F(\sigma) \exp \left(\int_{A \cap (0, 1)} s \frac{E'(s)}{E(s)} ds \right),$$

Lemma 26 implies that

$$\sigma \mapsto e^{c_3 \sigma} \frac{\sigma G(\sigma)}{I(\sigma, Q_\sigma)}$$

is non-decreasing. Since

$$\lim_{\sigma \to 0} \frac{E(\sigma)}{G(\sigma)} = \lim_{\sigma \to 0} \exp \left(- \int_{A \cap (0, \sigma)} s \frac{E'(s)}{E(s)} ds \right) \leq 1,$$

we have

$$\alpha = \lim_{\sigma\to 0} \frac{\sigma E(\sigma)}{I(\sigma, Q_\sigma)} \leq \lim_{\sigma\to 0} \frac{\sigma G(\sigma)}{I(\sigma, Q_\sigma)} = \lim_{\sigma\to 0} e^{c_3\sigma} \frac{\sigma G(\sigma)}{I(\sigma, Q_\sigma)}.$$

Therefore, we obtain that

$$\alpha \leq e^{c_3\sigma} \frac{\sigma G(\sigma)}{I(\sigma, Q_\sigma)} = e^{c_3\sigma} \frac{\sigma E(\sigma)}{I(\sigma, Q_\sigma)} \cdot \exp\left(\int_{A\cap(0,\sigma)} s\frac{E'(s)}{E(s)} ds \right) \tag{32}$$

and

$$\frac{\sigma E(\sigma)}{I(\sigma, Q_\sigma)} \leq e^{c_3\sigma} \frac{\sigma G(\sigma)}{I(\sigma, Q_\sigma)} \leq e^{\sigma_3} \frac{G(1)}{I(1)} =: K. \tag{33}$$

Now by the proof of Lemma 27, if $\theta_0^{n+1} \leq \sigma < \theta_0^n$, then

$$\begin{aligned} \int_{A\cap(0,\sigma)} s\frac{E'(s)}{E(s)} ds &\leq \int_0^\sigma s\frac{J'(s)}{J(s)} ds \\ &\leq \int_0^{\theta_0^n} s\frac{J'(s)}{J(s)} ds \\ &\leq \sum_{k=n}^{\infty} \int_{{\theta_0}^{k+1}}^{\theta_0^k} s\frac{J'(s)}{J(s)} ds \\ &\leq \log \theta_0^{-j\sum_{k=n}^{\infty} {\theta_0}^k} \\ &= \log \theta_0^{-\frac{j{\theta_0}^n}{1-\theta_0}} \\ &\leq c_4\theta_0^n \\ &\leq \frac{c_4}{\theta_0}\sigma =: c_5\sigma. \end{aligned} \tag{34}$$

Thus, this implies that for any $c_0 \geq 1$ and for

$$\tilde{E}(\sigma) := E(\sigma) \exp\left(c_0 \int_{A\cap(0,\sigma)} s\frac{E'(s)}{E(s)} ds \right)$$

$$E(\sigma) \leq \tilde{E}(\sigma) \leq e^{c_0 c_5 \sigma} E(\sigma), \tag{35}$$

and by (32)

$$\alpha \leq e^{(c_3+c_5)\sigma} \frac{\sigma E(\sigma)}{I(\sigma, Q_\sigma)}. \tag{36}$$

Furthermore, (15) and (19) imply that

$$\begin{aligned} 2E(\sigma) &\leq \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f,Q) d\Sigma_g + I(\sigma) + k\sigma^2 E'(\sigma) \\ &\leq I'(\sigma) - \frac{n-1}{\sigma} I(\sigma) + (1+c_2) I(\sigma) + k\sigma^2 E'(\sigma) \end{aligned} \tag{37}$$

where the constants c_2 and k depend on the Lipschitz bound of g. Inequality (37) implies

$$\begin{aligned}\frac{\frac{2\sigma E(\sigma)}{I(\sigma)}+n-1-O(\sigma)}{\sigma} &\leq \frac{I'(\sigma)}{I(\sigma)}+k\sigma^2\frac{E'(\sigma)}{I(\sigma)}\\ &\leq \frac{I'(\sigma)}{I(\sigma)}+kK\sigma\frac{E'(\sigma)}{E(\sigma)}\\ &\leq \frac{G'(\sigma)}{G(\sigma)}+\frac{1}{\sigma}+kK\sigma\frac{E'(\sigma)}{E(\sigma)}. \end{aligned}\tag{38}$$

for any $Q \in Y$. By combining (38) with (36) and absorbing the exponential terms in $O(\sigma)$ we obtain

$$\frac{2\alpha+n-1-O(\sigma)}{\sigma} \leq \frac{G'(\sigma)}{G(\sigma)}+\frac{1}{\sigma}+kK\sigma\frac{E'(\sigma)}{E(\sigma)}. \tag{39}$$

If $c_0 \geq kK+1$, then

$$\frac{\tilde{E}'(\sigma)}{\tilde{E}(\sigma)} \geq (1+c_0\sigma)\frac{E'(\sigma)}{E(\sigma)} \geq (1+(kK+1)\sigma)\frac{E'(\sigma)}{E(\sigma)} = \frac{G'(\sigma)}{G(\sigma)}+kK\sigma\frac{E'(\sigma)}{E(\sigma)}.$$

Therefore, using (39), we can choose c_0 sufficiently large so that

$$\frac{2\alpha+n-2}{\sigma}-c_0 \leq \frac{\tilde{E}'(\sigma)}{\tilde{E}(\sigma)}$$

and hence

$$\sigma \mapsto e^{c_0\sigma}\frac{\tilde{E}(\sigma)}{\sigma^{2\alpha+n-2}}$$

is non-decreasing. Q.E.D.

By combining Lemma 26 and Corollary 30 we obtain

Corollary 31 *Let* $\mathbf{B}$ *be a dimension-*n*, codimension-*ν *local model,* g *a normalized Lipschitz Riemannian metric defined on* $\mathbf{B}(r)$*,* (Y,d) *an NPC space,* $f:(\mathbf{B}(r),g)\to Y$ *a harmonic map and* $\alpha = ord^f(0)$*. Then there exist constants* k *and* $\sigma_0 \leq 1$ *depending on* $\mathbf{B}(r)$*, the Lipschitz bound and the ellipticity constant of* g *so that*

$$\frac{\sigma E(\sigma)}{I(\sigma,Q_\sigma)} \leq e^{k\rho}\frac{\rho E(\rho)}{I(\rho,Q_\rho)} \quad \text{for } 0<\sigma\leq\rho\leq\sigma_0 \tag{40}$$

and

$$\frac{E(\sigma)}{\sigma^{2\alpha+n-2}} \leq e^{k\rho}\frac{E(\rho)}{\rho^{2\alpha+n-2}} \quad \text{for } 0<\sigma\leq\rho\leq\sigma_0. \tag{41}$$

PROOF. Formula (40) immediatly follows from Lemma 26. Furthermore, Lemma 30 and (35) immediately imply (41). Q.E.D.

So far in this section, we assumed that our metric g is normalized at 0 (i.e. $g_{ij}(0) = \delta_{ij}$). We will relax this assumption and still show the monotonicity formula for the energy of a harmonic map (cf. Proposition 32 below). Let $\mathbf{B}$ be a dimension-n, codimension-$(n-k)$ local model and g a Lipschitz metric on $\mathbf{B}(r)$ with ellipticity constant $\lambda \in (0,1]$. For $x \in \mathbf{B}(r)$, recall that $R(x)$ is defined to be the radius of the largest homogeneous ball centered at x contained in $\mathbf{B}(r)$. Assume that x is a codimension-$(n-j)$ singular point. Let $\mathbf{B}'$ be a dimension-n, codimension-$(n-j)$ local model and $L_x : \mathbf{B}'(\lambda R(x)) \to L_x(\mathbf{B}'(\lambda R(x))) \subset \mathbf{B}$ be a homeomorphism satisfying properties (i) through (iv) of Proposition 6. In particular, recall this implies that $h := L_x^* g$ is a normalized metric. If $f : \mathbf{B}(r) \to Y$ is any finite energy map, then $f \circ L_x$ is defined on $\mathbf{B}'(\lambda R(x))$. Moreover,

$$\int_{B_x(\sigma)} |\nabla f|^2 d\mu_g = \int_{L_x^{-1}(B_x(\sigma))} |\nabla (f \circ L_x)|^2 d\mu_h \tag{42}$$

and

$$\int_{\mathbf{B}'(\sigma)} |\nabla (f \circ L_x)|^2 d\mu_h = \int_{L_x(\mathbf{B}'(\sigma))} |\nabla f|^2 d\mu_g. \tag{43}$$

This in turn implies that if f is a harmonic map with respect to the metric g, then $f \circ L_x$ is a harmonic map with respect to the metric h. We call $f \circ L_x$ the normalized harmonic map at x. Recall that σ_0 was defined above as the upper bound for which monotoncity formulae of Lemma 24 and Corollary 31 are valid for any harmonic map from a local model with a normalized metric. Therefore, these monotonicity formulae for $f \circ L_x$ are valid for balls $\mathbf{B}'(\sigma)$ contained in $\mathbf{B}'(r_0(x))$ for

$$r_0(x) := \min\{\sigma_0, \lambda R(x)\}. \tag{44}$$

For a harmonic map f, we define the *order of f at x* as

$$ord^f(x) := ord^{f \circ L_x}(0).$$

We also define $E_x(\sigma)$ and $I_x(\sigma)$ for σ sufficiently small by setting

$$E_x(\sigma) = \int_{B_x(\sigma)} |\nabla f|^2 d\mu_g$$

and

$$I_x(\sigma) = \int_{\partial B_x(\sigma)} d^2(f, Q) d\Sigma_g.$$

Proposition 32 *Let $\mathbf{B}$ be a dimension-n, codimenion-$(n-k)$ local model, g a Lipschitz metric defined on $\mathbf{B}(r)$ with ellipticity constant $\lambda \in (0,1]$, (Y,d) an NPC space and $f : (\mathbf{B}(r), g) \to Y$ a harmonic map. Then there exist constants*

$\gamma > 0$ and $C \geq 1$ depending on $\mathbf{B}(r)$, the Lipschitz bound and the ellipticity constant of g so that for every $x \in \mathbf{B}(r)$,

$$\frac{E_x(\sigma)}{\sigma^{n-2+2\gamma}} \leq C \frac{E_x(\rho)}{\rho^{n-2+2\gamma}},\ 0 < \sigma < \rho \leq r(x) \tag{45}$$

and

$$\frac{E_x(\sigma)}{\sigma^{n-2+2\alpha_x}} \leq C \frac{E_x(\rho)}{\rho^{n-2+2\alpha_x}},\ 0 < \sigma < \rho \leq r(x) \tag{46}$$

where

$$r(x) = \lambda r_0(x) = \min\{\lambda\sigma_0, \lambda^2 R(x)\}. \tag{47}$$

Here, recall that $R(x)$ is the radius of the largest homogeneous ball contained in $St(x)$ and $\sigma_0 > 0$ defined in Corollary 31 is the number associated with the monotonicity formulae.

Proof. Let L_x be as above and set

$$\mathcal{E}(\sigma) = \int_{\mathbf{B}'(\sigma)} |\nabla(f \circ L_x)|^2 d\mu_h.$$

Lemma 24 and Corollary 31 imply that there exists a constant $c \geq 1$ so that

$$\frac{\mathcal{E}(s)}{s^{n-2+2\gamma}} \leq c \frac{\mathcal{E}(r)}{r^{n-2+2\gamma}},\ 0 < s < r \leq r_0(x) \tag{48}$$

and

$$\frac{\mathcal{E}(s)}{s^{n-2+2\alpha_x}} \leq c \frac{\mathcal{E}(r)}{r^{n-2+2\alpha_x}},\ 0 < s < r \leq r_0(x) \tag{49}$$

with $r_0(x)$ as in (44). Let $\Gamma = n-2+2\gamma$ or $\Gamma = n-2+2\alpha_x$. Fix σ, ρ so that $0 < \sigma < \rho \leq r(x)$. Then, since $\lambda \leq 1$, $0 < \lambda^{-1}\sigma, \lambda\rho \leq r_0(x)$. We prove (45) and (46) by considering the following two cases. In the first case, we assume $\lambda^{-1}\sigma \leq \lambda\rho$. We then have

$$\begin{aligned}
\frac{E_x(\sigma)}{\sigma^\Gamma} &\leq \frac{\mathcal{E}(\lambda^{-1}\sigma)}{\sigma^\Gamma} \quad \text{by (42) and the fact that } L_x^{-1}(B_x(\sigma)) \subset \mathbf{B}'(\lambda^{-1}\sigma) \\
&\leq \frac{1}{\lambda^\Gamma} \frac{\mathcal{E}(\lambda^{-1}\sigma)}{(\lambda^{-1}\sigma)^\Gamma} \\
&\leq \frac{c}{\lambda^\Gamma} \frac{\mathcal{E}(\lambda\rho)}{(\lambda\rho)^\Gamma} \quad \text{by (48) or (49) and the assumption that } \lambda^{-1}\sigma \leq \lambda\rho \\
&\leq \frac{c}{\lambda^\Gamma} \frac{E_x(\rho)}{(\lambda\rho)^\Gamma} \quad \text{by (43) and the fact that } L_x(\mathbf{B}'(\lambda\rho)) \subset B_x(\rho) \\
&\leq \frac{c}{\lambda^{2\Gamma}} \frac{E_x(\rho)}{\rho^\Gamma}.
\end{aligned}$$

In the second case, we assume $\lambda^{-1}\sigma > \lambda\rho$. We then have

$$\begin{aligned} \frac{E_x(\sigma)}{\sigma^\Gamma} &\leq \frac{E_x(\rho)}{\sigma^\Gamma} \text{ by the fact that } B_x(\sigma) \subset B_x(\rho) \\ &\leq \frac{E_x(\rho)}{(\lambda^2\rho)^\Gamma} \text{ by the fact that } \tfrac{1}{\sigma} < \tfrac{1}{\lambda^2\rho} \\ &\leq \frac{1}{\lambda^{2\Gamma}} \frac{E_x(\rho)}{\rho^\Gamma}. \end{aligned}$$

In either case, we have proven our assertion by setting $C = \frac{c}{\lambda^{2\Gamma}}$. Q.E.D.

4 Hölder continuity

In this section, we prove the Hölder continuity of a harmonic map from a Riemannian complex into an NPC space (Y, d). Such a result in the case when the domain metric is smooth was discussed in [EF] and [Ch]. Using the results of the previous section, we are able to consider a Lipschitz metric g. Moreoever, we provide the explicit dependence of the Hölder exponent and Hölder constant on g, E^f and $\mathbf{B}$. By dependence on $\mathbf{B}$, we mean the dependence on the dimension of $\mathbf{B}$, the number of wedges as well as the wedge angles of $\mathbf{B}$. In the later sections, we give a condition for which the Hölder continuity can be improved to Lipschitz continuity. Our proof follows the approach in [GS] and [Ch]. The main technical difficulty is that monotonicity only works for small balls (cf. Remark 3.3 [DM1]). Therefore in order to obtain the energy decay estimate for large balls (cf. Proposition 37), we need the rather technical inductive process described in Proposition 35 and Corollary 36. We first prove some results pertaining to the geometry of local models.

Proposition 33 *Fix integers k, n so that $0 \leq k < n$. Assume that the sets $\mathcal{B}_k, ..., \mathcal{B}_n$ have the following properties:*

(1) For each $j \in \{k, k+1, ..., n\}$, $\mathcal{B}_j$ is a finite set of dimension-n, codimension-$(n-j)$ local models, and

(2) if $\mathbf{B} \in \mathcal{B}_j$ for some $j \in \{k, k+1, ..., n-1\}$, then for any $x \in \mathbf{B}$ and $\sigma \leq R(x)$, we have $B_x(\sigma)$ is isometric to $\mathbf{B}'(\sigma)$ where $\mathbf{B}' \in \mathcal{B}_i$ for some $i \in \{j, j+1, ..., n\}$.

Then for all $j \in \{k, ..., n\}$ and $\mathbf{B} \in \mathcal{B}_j$, there exists $\kappa(\mathbf{B}) \geq 1$ so that for all $i \in \{j, ..., n\}$,

$$\frac{|x - \bar{x}|}{R(x)} < \kappa(\mathbf{B}), \ \forall x \in S_i - S_{i-1} \subset \mathbf{B}, \forall \bar{x} \in \pi_{i-1}(x).$$

Proof. We first make the following observation. Let $\mathbf{B}$ be any dimension-n, codimension-$(n-k)$ local model and $x \in \mathbf{B}$. Recall that $D = S_k$ is isometric to $\mathbf{R}^k$ and hence the closest point projection map $\pi_D : \mathbf{B} \to D$ is well-defined. For any $x \in \mathbf{B} - D$, let $t \mapsto x_t$ be the constant speed parameterization of a ray starting from $\pi_D(x)$ and going through $x = x_1$. Assume $x \in S_i - S_{i-1}$ and let $\bar{x} \in \pi_{i-1}(x)$. Since $t \mapsto x_t$ and $t \mapsto \overline{x}_t$ are rays from $\pi_D(\overline{x}) = \pi_D(x)$, we see that $\overline{x}_t \in \pi_{i-1}(x_t)$ and $t|x - \overline{x}| = |x_t - \overline{x}_t|$. Furthermore, we also see that $tR(x) = R(x_t)$. Thus, we observe that

$$\frac{|x - \bar{x}|}{R(x)} = \frac{|x_t - \overline{x}_t|}{R(x_t)}, \ \forall t \in (0, \infty). \tag{50}$$

We now proceed with the proof of the assertion by reverse induction on j. First note that there is nothing to prove for $j = n$ since $\mathcal{B}_n = \{\mathbf{R}^n\}$. Assume that the assertion is true for $\mathcal{B}_n, \mathcal{B}_{n-1}, ..., \mathcal{B}_{j+1}$. By (50), we only need to show that for each $\mathbf{B} \in \mathcal{B}_j$, there exists $\kappa(\mathbf{B})$ so that for any $i \in \{j, ..., n\}$,

$$\frac{|x - \bar{x}|}{R(x)} < \kappa(\mathbf{B}), \ \forall x \in U \cap (S_i - S_{i-1}) \subset \mathbf{B}, \forall \bar{x} \in \pi_{i-1}(x)$$

where U is the set of points of $\mathbf{B}$ at a distance 1 from D. Suppose this is not true, i.e. for some fixed i, there exist $\mathbf{B} \in \mathcal{B}_j$ and a sequence $y_n \in U \cap (S_i - S_{i-1})$ so that

$$\frac{|y_n - \overline{y_n}|}{R(y_n)} \to \infty \tag{51}$$

with $\overline{y_n} \in \pi_{i-1}(y_n)$. Since U is a compact set, we may assume (by choosing a subsequence if necessary) that $y_n \to y$. By the definition of U, $y \in S_m - S_{m-1}$ for $m > j$. Let's also assume $i > j + 1$. By the facts that $y \in S_m - S_{m-1}$, $|y_n - y| \to 0$ and assumption (2), we can assume that y_n and $\overline{y_n}$ are points in a local model $\mathbf{B}' \in \mathcal{B}_m$. Since $m > j$ then the inductive hypothesis implies that $\frac{|y_n - \overline{y_n}|}{R(y_n)}$ is bounded which contradicts (51). Now consider the case $i = j + 1$, hence $m = j + 1$. Since in this case $|y_n - \overline{y_n}| = 1$ and because $U \cap S_{j+1}$ is compact and hence $R(y_n) \geq c$ independent of n, we also obtain a contradiction to (51). This completes the proof. Q.E.D.

Corollary 34 *Let $\mathbf{B}$ be a dimension-n, codimension-$(n-k)$ local model. There exists $\kappa \geq 1$ so that for any $i = \{k, ..., n\}$,*

$$\frac{|x - \pi_{i-1}(x)|}{R(x)} < \kappa, \ \forall x \in S_i - S_{i-1} \subset \mathbf{B}.$$

Proof. Apply Proposition 35 with $\mathcal{B}_j$ for $j = k, .., n$ defined to be the set of model spaces so that $\mathbf{B}' \in \mathcal{B}_j$ if and only if there exists $x \in S_j - S_{j-1}$ so that $B_x(R(x))$ is isometric to $\mathbf{B}'(R(x))$. We are done by setting $\kappa = \max \kappa(\mathbf{B})$ where the maximum is taken over $\mathbf{B} \in \mathcal{B}_j$ and $j = k, \ldots, n$. Q.E.D.

Proposition 35 *Fix $\lambda \in (0,1]$ and integers k, n so that $0 \leq k < n$. Assume that the sets $\mathcal{B}_k, ..., \mathcal{B}_n$ have the following properties:*

(1) For each $j \in \{k, k+1, ..., n\}$, $\mathcal{B}_j$ is a finite set of dimension-n, codimension-$(n-j)$ local models, and

(2) if $\mathbf{B} \in \mathcal{B}_j$ for some $j \in \{k, k+1, ..., n-1\}$, then for any $x \in \mathbf{B}$ and $\sigma \leq R(x)$, we have $B_x(\sigma)$ is isometric to $\mathbf{B}'(\sigma)$ where $\mathbf{B}' \in \mathcal{B}_i$ for some $i \in \{j, j+1, ..., n\}$.

Then, there exists $C \geq 1$ so that for any $\mathbf{B} \in \mathcal{B}_j$, $j \in \{k, k+1, ..., n\}$, and $x \in \mathbf{B}$, there exist an ordered sequence

$$x_1 \rhd x_2 \rhd ... \rhd x_m$$

of points in $\mathbf{B}$ with $x_1 = x$, $x_m \in S_j$ and positive numbers $\sigma_1, ..., \sigma_{m-1}$ with the property that

$$\frac{\sigma_i}{R(x_i)} \leq C, \; \frac{\sigma_i}{R(x_{i+1})} \leq 1 \quad \textit{and} \quad B_{x_i}(R(x_i)) \subset B_{x_{i+1}}(\lambda^2 \sigma_i). \tag{52}$$

Proof. We first need some preliminary constructions on each element $\mathbf{B}$ of $\cup_{j=k}^{n} \mathcal{B}_j$. So fix j and $\mathbf{B} \in \mathcal{B}_j$. Let U be the set of points of $\mathbf{B}$ at a distance 1 from $D = S_j$. Set

$$U_n = U, \; U_{n-1} = \pi_{n-1}(U_n), \; \ldots \; , \; U_j = \pi_j(U_{j+1}).$$

By the convexity of the faces of $\mathbf{B}$, we see that $U_{j+1} \subset\subset \mathbf{B} - S_j$.

For $i = j+1, ..., n-1$, we define a positive number R_i and a subset N_i of $\mathbf{B}$ by an inductive procedure.

- First we define R_{j+1} and N_{j+1}.

Let V_{j+1} be so that

$$U_{j+1} \subset\subset V_{j+1} \subset\subset S_{j+1} - S_j.$$

Thus, there exists $R_{j+1} > 0$ so that

$$R(x') \geq R_{j+1}, \; \forall x' \in V_{j+1}.$$

We can choose a neighborhood $N_{j+1} \subset\subset \mathbf{B} - S_j$ of U_{j+1} so that

$$B_{x'}(2\sigma') \subset \mathbf{B} - S_j, \forall x' \in \Pi_{j+1}(x) \text{ where } x \in N_{j+1} \text{ and } \sigma' = |x - x'|,$$

$$\Pi_{j+1}(N_{j+1}) \subset V_{j+1}$$

and

$$U_{j+1} \subset\subset N_{j+1}.$$

• Assuming we have chosen positive numbers $R_{j+1},...,R_{i-1}$, open sets $N_{j+1} \subset\subset \mathbf{B}-S_j,...,N_{i-1} \subset\subset \mathbf{B}-S_{i-2}$ and sets $V_{j+1},...,V_{i-1}$ so that for $l \in \{j+1,...,i-1\}$,

$$B_{x'}(2\sigma') \subset \mathbf{B} - S_{l-1},\ \forall x' \in \Pi_l(x) \text{ where } x \in N_l \text{ and } \sigma' = |x - x'|,$$

$$\Pi_l(N_l) \subset V_l,$$

$$U_l - \cup_{m=j+1}^{l-1} N_m \subset\subset N_l,$$

and

$$R(x') \geq R_l,\ \forall x' \in V_l,$$

we define R_i and N_i as follows:

First note that

$$U_i - \cup_{m=j+1}^{i-1} N_m \subset\subset U_i - S_{i-1},$$

hence, we can choose $V_i \subset S_i$ be so that

$$U_i - \cup_{m=j+1}^{i-1} N_m \subset\subset V_i \subset\subset U_i - S_{i-1}.$$

Thus, there exists $R_i > 0$ so that

$$R(x') \geq R_i,\ \forall x' \in V_i.$$

We can choose a neighborhood $N_i \subset\subset \mathbf{B} - S_{i-1}$ of $U_i - \cup_{l=j+1}^{i-1} N_l$ so that

$$B_{x'}(2\sigma') \subset \mathbf{B} - S_{i-1},\ \forall x' \in \Pi_i(x) \text{ where } x \in N_i \text{ and } \sigma' = |x - x'|,$$

$$\Pi_i(N_i) \subset V_i$$

and

$$U_i - \cup_{l=j+1}^{i-1} N_l \subset\subset N_i.$$

In summary, we have constructed sets $U = U_n,...,U_{j+1},\ V_n,\ldots,V_{j+1}$, positive numbers $R_{j+1},...,R_{n-1}$ and open sets $N_{j+1} \subset\subset \mathbf{B} - S_j,...,N_{n-1} \subset\subset \mathbf{B} - S_{n-2}$ so that for each $l = j+1,...,n-1$,

$$B_{x'}(2\sigma') \subset \mathbf{B} - S_{l-1}$$

for $x \in N_l, x' \in \Pi_l(x), \sigma' = |x - x'|$,

$$\Pi_l(N_l) \subset V_l$$

and

$$R(x') \geq R_l, \forall x' \in V_l.$$

By assumption 2, and by shrinking $N_{j+1}, \ldots, N_{n-1}$ if necessary, we assume the following for $x \in N_l, x' \in \Pi_l(x), \sigma' = |x - x'|$:

$$\text{we can identify } B_{x'}(\sigma') \text{ with } \mathbf{B}'(\sigma') \text{ where } \mathbf{B}' \in \mathcal{B}_l. \tag{53}$$

Since

$$N = \cup_{l=j+1}^{n-1} N_l$$

covers the singular set of U, there exists $R_n > 0$ so that

$$R(x) \geq R_n, \ \forall x \in U - N.$$

In the above, for each $j \in \{k, ..., n\}$ and $\mathbf{B} \in \mathcal{B}_j$, we associated sets

$$U = U(\mathbf{B}), U_{n-1} = U_{n-1}(\mathbf{B}), ..., U_{j+1} = U_{j+1}(\mathbf{B}),$$

positive numbers

$$R_{j+1} = R_{j+1}(\mathbf{B}), ..., R_n = R_n(\mathbf{B})$$

and open sets

$$N_{j+1} = N_{j+1}(\mathbf{B}), ..., N_{n-1} = N_{n-1}(\mathbf{B}), N = N(\mathbf{B}).$$

Let

$$C(\mathbf{B}) = \max\{\frac{2}{R_{j+1}(\mathbf{B})}, ..., \frac{2}{R_n(\mathbf{B})}, 1\},$$

and

$$C = \lambda^{-2} \max\{C(\mathbf{B}) : \mathbf{B} \in \mathcal{B}_j, j = k, ..., n\}.$$

Furthermore, for $l = k, ..., n$, let

$$\hat{R}_l = \min\{R_l(\mathbf{B}) : \mathbf{B} \in \mathcal{B}_j, j = k, ..., n\}.$$

We now proceed with the proof of the proposition. Since $\mathcal{B}_n = \{\mathbf{R}^n\}$, there is nothing to prove for $\mathcal{B}_n$. We now prove the assertion for $\mathcal{B}_j$ for any $j = k, ..., n$ by doing a reverse induction; more specifically, assume that the assertion is true for $\mathcal{B}_n$, $\mathcal{B}_{n-1}, ..., \mathcal{B}_{j+1}$ and prove the assertion for $\mathcal{B}_j$.

Now given $x \in \mathbf{B} \in \mathcal{B}_j$, we need to show that there exist an ordered sequence

$$x_1 \triangleright x_2 \triangleright ... \triangleright x_m$$

of points in $\mathbf{B}$ with $x_1 = x$, $x_m \in S_j$ and positive numbers $\sigma_1, ..., \sigma_{m-1}$ satisfying (52); i.e.

$$\frac{\sigma_i}{R(x_i)} \leq C, \ \frac{\sigma_i}{R(x_{i+1})} \leq 1 \ \text{ and } \ B_{x_i}(R(x_i)) \subset B_{x_{i+1}}(\lambda^2 \sigma_i).$$

If x is in the lowest dimensional stratum $D = S_j$, there is nothing to prove so assume $x \in \mathbf{B} - S_j$. By the scale invariance of the assertion, we may assume that

$x \in U(\mathbf{B})$. If $x \in U(\mathbf{B}) - N(\mathbf{B})$, then let $x_1 = x$, $x_2 \in \Pi_j(x_1)$ and $\sigma_1 = 2\lambda^{-2}$. Since

$$\frac{\sigma_1}{R(x_1)} = \frac{2\lambda^{-2}}{R(x_1)} \leq \frac{2\lambda^{-2}}{\hat{R}_n} \leq C,$$

$R(x_1) \leq 1$ and $R(x_2) = \infty$, we are done.

So assume $x \in N(\mathbf{B})$; in particular, $x \in N_l(\mathbf{B})$ for some $l = j+1, ..., n-1$. In this case, we use the inductive hypothesis. More specifically, choose $x' \in \Pi_l(x)$, let $\sigma' = |x - x'|$, use (53) and note that the inductive hypothesis implies that for any $\mathbf{B}' \in \mathcal{B}_l$ and any $x \in \mathbf{B}'$, there there exist a sequence

$$x_1 = x \rhd x_2 \rhd ... \rhd x_m \in S_l$$

and $\sigma_1, ..., \sigma_{m-1}$ with the property that

$$\frac{\sigma_i}{R(x_i)} \leq C_l(\mathbf{B}') \leq C, \quad \frac{\sigma_i}{R(x_{i+1})} \leq 1$$

and

$$B_{x_i}(R(x_i)) \subset B_{x_{i+1}}(\lambda^2 \sigma_i).$$

Since $x \in U(\mathbf{B})$, we have that $x_m \in U_l(\mathbf{B}) = \Pi_l(U(\mathbf{B}))$. Thus, $R(x_m) \geq R_l(\mathbf{B}) \geq \hat{R}_l$. Therefore, if we set $\sigma_m = 2\lambda^{-2}$, then

$$\frac{\sigma_m}{R(x_{m+1})} \leq 1$$

since $R(x_{m+1}) = \infty$ and

$$\frac{\sigma_m}{R(x_m)} = \frac{2\lambda^{-2}}{R(x_m)} \leq \frac{2\lambda^{-2}}{\hat{R}_l} \leq C$$

Furthermore, since the distance of x to S_j is equal to 1, $R(x_m) \leq 1$. Hence

$$B_{x_m}(R(x_m)) \subset B_{x_{m+1}}(\lambda^2 \sigma_m).$$

This completes the inductive step and finishes the proof of Proposition 35. Q.E.D.

Corollary 36 *Let* $\mathbf{B}$ *be a dimension-*n*, codimension-*$(n-k)$ *local model,* g *a Lipschitz Riemannian metric defined on* $\mathbf{B}(r)$*,* (Y, d) *an NPC space and* $f : (\mathbf{B}(r), g) \to Y$ *a finite energy map. Fix* $\varrho \in (0,1)$*. Then there exist* $C \geq 1$ *and* $R_0 > 0$ *depending on* g*,* $\mathbf{B}(r)$ *and* ϱ *so that for every* $x \in \mathbf{B}(\varrho r)$*, there exist a sequence of points*

$$x = x_1 \rhd ... \rhd x_m$$

and a sequence of positive numbers

$$\sigma_1, ..., \sigma_{m-1}$$

so that for $i = 1, ..., m-1$,

$$\frac{\sigma_i}{r(x_i)} \leq C, \ \frac{\sigma_i}{r(x_{i+1})} \leq 1, \ E_{x_i}(r(x_i)) \leq E_{x_{i+1}}(\sigma_i) \tag{54}$$

and

$$r(x_m) \geq R_0 \tag{55}$$

where $r(x) = \min\{\lambda\sigma_0, \lambda^2 R(x)\}$ *is as defined in (47).*

PROOF. We define $\mathcal{B}_j$ for $j = k, .., n$ to be the set of model spaces so that $\mathbf{B}' \in \mathcal{B}_j$ if and only if there exists $x \in S_j - S_{j-1}$ so that $B_x(R(x))$ is isometric to $\mathbf{B}'(R(x))$. Then $\mathcal{B}_k, ..., \mathcal{B}_n$ satisfy conditions (1) and (2) of Proposition 35.

Let $y \in \mathbf{B}(\varrho r)$ and assume $y \in S_j - S_{j-1}$. Choose $r_y > 0$ sufficiently small so that $B_y(r_y)$ is compactly supported away from S_{j-1} and $r_y << \sigma_0$. In this way, we see that $\lambda^2 R(y') = r(y')$ for $y' \in B_y(r_y)$ and $R_y := \inf\{r(y') : y' \in B_y(r_y) \cap S_j\} > 0$. Choose a finite covering $\{B_{y_l}(r_{y_l}) : l = 1, ..., N\}$ of $\overline{\mathbf{B}\left(\frac{r}{2}\right)}$. Thus, given $x \in \mathbf{B}(\varrho r)$, there exists $l \in \{1, ..., N\}$ so that $x \in B_{y_l}(r_{y_l})$. By construction, the point y_l is an element of S_j for some $j \in \{k, ..., n\}$ and $B_{y_l}(r_{y_l})$ is isometric to $\mathbf{B}(r_{y_l})$ for some $\mathbf{B} \in \mathcal{B}_j$. Applying Proposition 35 and noting that $r = \lambda^2 R$, we obtain sequences $x_1 \rhd ... \rhd x_m$ and $\sigma'_1, ..., \sigma'_{m-1}$ which satisfy

$$\frac{\sigma'_i}{\lambda^{-2} r(x_i)} \leq C, \ \frac{\sigma'_i}{\lambda^{-2} r(x_{i+1})} \leq 1 \ \text{ and } \ B_{x_i}(\lambda^{-2} r(x_i)) \subset B_{x_{i+1}}(\lambda^2 \sigma'_i)$$

by (52). If we set $\sigma_i = \lambda^2 \sigma'_i$, we obtain

$$\frac{\sigma_i}{r(x_i)} \leq C, \ \frac{\sigma_i}{r(x_{i+1})} \leq 1 \text{ and } \ B_{x_i}(\lambda^{-2} r(x_i)) \subset B_{x_{i+1}}(\sigma_i).$$

Since $\lambda \in (0, 1]$, the inclusion above shows that

$$B_{x_i}(r(x_i)) \subset B_{x_{i+1}}(\sigma_i)$$

which in turn implies

$$E_{x_i}(r(x_i)) \leq E_{x_{i+1}}(\sigma_i).$$

Finally, if we set $R_0 = \min\{R_{y_1}, .., R_{y_N}\}$, then we obtain $r(x_m) \geq R_0 > 0$. Q.E.D.

Proposition 37 *Let* $\mathbf{B}$ *be a dimension-*n*, codimension-*$(n-k)$ *local model,* g *a Lipschitz Riemannian metric defined on* $\mathbf{B}(r)$*,* (Y, d) *a metric space and* $f : (\mathbf{B}(r), g) \to Y$ *a finite energy map. Fix* $\varrho \in (0, 1)$ *and suppose that for* $x \in \mathbf{B}(\varrho r)$ *there exist* $\beta > 0$ *and* $\hat{C} \geq 1$ *so that*

$$\frac{E_x(\sigma)}{\sigma^{n-2+2\beta}} \leq \hat{C} \frac{E_x(\rho)}{\rho^{n-2+2\beta}}, \quad 0 < \sigma \leq \rho \leq r(x) \tag{56}$$

where $r(x) = \min\{\lambda\sigma_0, \lambda^2 R(x)\}$ *is as defined in (47). Then there exist* K *and* $R > 0$ *depending only on the total energy* E^f *of* f*, the ellipticity constant* $\lambda \in (0,1]$*, the Lipschitz bound of* g*,* $\mathbf{B}(r)$ *and* ϱ *so that*

$$E_x(\sigma) \leq K^2 \sigma^{n-2+2\beta}, \quad \forall x \in \mathbf{B}(\varrho r), \sigma < R.$$

Proof. Set

$$R := \min\{R_0, \sigma_0\}$$

where $R_0 > 0$ is as in Corollary 36 and $\sigma_0 \in (0,1]$ as in Corollary 31,

$$K_0 := \left(\hat{C}C^{n-2+2\beta}\right)^n \frac{E^f}{R^{n-2+2\beta}}$$

where $C \geq 1$ as in Corollary 36 and

$$K := \left(\frac{2^n \kappa^n}{\lambda^4 \sigma_0}\right)^{\frac{n}{2}-1+\beta} \max\{\sqrt{K_0}, \sqrt{E^f}\}.$$

where $\kappa \geq 1$ is defined in Corollary 34. Let $x \in \mathbf{B}(\varrho r)$ and $\sigma < R$.

Case 1. Assume that $\sigma \leq r(x)$. By (56),

$$\frac{E_x(\sigma)}{\sigma^{n-2+2\beta}} \leq \hat{C} \frac{E_x(r(x))}{r(x)^{n-2+2\beta}}.$$

Let $x = x_1 \rhd \ldots \rhd x_m$ and $\sigma_1, \ldots, \sigma_{m-1}$ be as in Corollary 36. By (54) and (56),

$$\begin{aligned} \frac{E_{x_i}(r(x_i))}{r(x_i)^{n-2+2\beta}} &\leq \frac{E_{x_{i+1}}(\sigma_i)}{r(x_i)^{n-2+2\beta}} \\ &\leq C^{n-2+2\beta} \frac{E_{x_{i+1}}(\sigma_i)}{\sigma_i^{n-2+2\beta}} \\ &\leq \hat{C}C^{n-2+2\beta} \frac{E_{x_{i+1}}(r(x_{i+1}))}{r(x_{i+1})^{n-2+2\beta}} \end{aligned}$$

for $i = 1, \ldots, m-1$. Additionally, by (55), we have that

$$\frac{E_{x_m}(r(x_m))}{r(x_m)^{n-2+2\beta}} \leq \frac{E_{x_m}(r(x_m))}{R^{n-2+2\beta}} \leq \frac{E^f}{R^{n-2+2\beta}}$$

and hence

$$\frac{E_{x_1}(\sigma)}{\sigma^{n-2+2\beta}} \leq \left(\hat{C}C^{n-2+2\beta}\right)^m \frac{E^f}{R^{n-2+2\beta}}.$$

This implies

$$E_x(\sigma) \leq K^2 \sigma^{n-2+2\beta} \text{ whenever } \sigma \leq r(x).$$

Case 2. Alternately, assume $r(x) < \sigma$. Since $r(x) = \min\{\lambda\sigma_0, \lambda^2 R(x)\}$, either

$$\lambda\sigma_0 < \sigma \quad \text{or} \quad R(x) < \frac{\sigma}{\lambda^2}.$$

First consider the case when $\lambda\sigma_0 < \sigma$. Then

$$E_x(\sigma) \leq E_x(\sigma)\left(\frac{\sigma}{\lambda\sigma_0}\right)^{n-2+2\beta} \leq \frac{E^f}{(\lambda\sigma_0)^{n-2+2\beta}}\sigma^{n-2+2\beta} \leq K^2\sigma^{n-2+2\beta}.$$

Next assume $R(x) < \frac{\sigma}{\lambda^2}$. The fact that $R(x') = \infty$ for every $x' \in D = S_k$ implies that $x \notin D = S_k$. So let $i \in \{k+1, ..., n\}$ be so that $x \in S_i - S_{i-1}$. Furthermore, let $x = y_i, ..., y_k \in D = S_k$ where $y_{j-1} \in \Pi_{j-1}(y_j)$. We now follow the following finite step procedure.

Step 1. Since $R(y_i) < \frac{\sigma}{\lambda^2}$, there exists $l_1 \in \{k+1, ..., i-1\}$ so that

$$R(y_m) < \frac{2^{i-m}\sigma}{\lambda^2}, \ \forall m = l_1 + 1, ..., i.$$

Thus, we can apply Corollary 34 to obtain

$$|y_m - y_{m-1}| < \kappa R(y_m) < \frac{2^{i-m}\kappa}{\lambda^2}\sigma.$$

This implies

$$B_{y_i}\left(\frac{\kappa}{\lambda^2}\sigma\right) \subset B_{y_{i-1}}\left(\frac{2\kappa}{\lambda^2}\sigma\right) \subset ... \subset B_{y_{l_1}}\left(\frac{2^{i-l_1}\kappa}{\lambda^2}\sigma\right).$$

Since $\frac{\kappa}{\lambda^2} \geq 1$, we also have

$$B_{y_i}(\sigma) \subset B_{y_i}\left(\frac{\kappa}{\lambda^2}\sigma\right).$$

The above inclusions imply

$$E_x(\sigma) \leq E_{y_{l_1}}\left(\frac{2^{i-l_1}\kappa}{\lambda^2}\sigma\right). \tag{57}$$

Now we consider the two possibilities, either

$$\frac{2^{i-l_1}\kappa}{\lambda^2}\sigma \leq r(y_{l_1}) \quad \text{or} \quad r(y_{l_1}) < \frac{2^{i-l_1}\kappa}{\lambda^2}\sigma.$$

In the former possibility, we use (57) and Case 1 to see that

$$E_x(\sigma) \leq E_{y_{l_1}}\left(\frac{2^{i-l_1}\kappa}{\lambda^2}\sigma\right) \leq K_0^2\left(\frac{2^{i-l_1}\kappa}{\lambda^2}\sigma\right)^{n-2+2\beta} \leq K^2\sigma^{n-2+2\beta}.$$

In the latter possibility, either

$$\text{(A) } \sigma_0 < \frac{2^{i-l_1}\kappa}{\lambda^4}\sigma \text{ or (B)} \quad R(y_{l_1}) < \frac{2^{i-l_1}\kappa}{\lambda^4}\sigma.$$

In Case (A), we use (57) and the fact that $1 \le \frac{2^{i-l_1}\kappa}{\lambda^4\sigma_0}\sigma$ to obtain

$$E_x(\sigma) \le E_{y_{l_1}}\left(\frac{2^{i-l_1}\kappa}{\lambda^2}\sigma\right) \cdot \left(\frac{2^{i-l_1}\kappa}{\lambda^4\sigma_0}\right)^{n-2+2\beta} \sigma^{n-2+2\beta} \le K^2\sigma^{n-2+2\beta}.$$

If (B) is true, then we proceed to Step 2.

Step 2. Since we are assuming $R(y_{l_1}) < \frac{2^{i-l_1}\kappa}{\lambda^4}\sigma$, there exists $l_2 \in \{k+1, ..., l_1-1\}$ so that

$$R(y_m) < \frac{2^{i-m}\kappa}{\lambda^4}\sigma, \ \forall m = l_2+1, ..., l_1.$$

Thus, we can apply Corollary 34 to obtain

$$|y_m - y_{m-1}| < \kappa R(y_m) < \frac{2^{i-m}\kappa^2}{\lambda^4}\sigma, \ \forall m = l_2+1, ..., l_1.$$

Hence

$$B_{y_{l_1}}\left(\frac{2^{i-l_1}\kappa^2}{\lambda^4}\sigma\right) \subset ... \subset B_{y_{l_2}}\left(\frac{2^{i-l_2}\kappa^2}{\lambda^4}\sigma\right).$$

Combined with (57), this implies

$$E_x(\sigma) \le E_{y_{l_2}}\left(\frac{2^{i-l_2}\kappa^2}{\lambda^4}\sigma\right).$$

We now continue. In the similar way as in Step 1, we prove $E_x(\sigma) \le K^2\sigma^{n-2+2\beta}$ or we continue to Step 3 where we assume $R(y_{l_2}) < \frac{2^{i-l_2}\kappa^2}{\lambda^4}\sigma$. At Step S of this procedure, we produce an integer $l_S \in \{k, ..., i\}$ and this procedure terminates after a finite number of steps since $l_S > l_{S+1}$. Finally, observe that in order to prove our assertion, we must show that if N is the number of steps taken and if $y_{l_N} = y_k$ then the case corresponding to (B) (i.e. the case that $R(y_{l_N}) < \frac{2^{i-l_N}\kappa^N}{\lambda^4}\sigma$) does not occur. This is true because $R(y_{l_N}) = R(y_k) = \infty$ and this completes the proof. Q.E.D.

The following is a version of the Campanato Lemma (cf. [Si] Lemma 1).

Lemma 38 *Let* $\mathbf{B}$ *be a dimension-*n*, codimension-*$(n-k)$ *local model,* g *a Lipschitz Riemannian metric defined on* $\mathbf{B}(r)$*,* (Y, d) *an NPC space and* $f : (\mathbf{B}(r), g) \to Y$ *an* L^2*-map. Fix* $\varrho \in (0,1)$*. If there exist* $K > 0$*,* $R \in (0, (1-\varrho)r)$ *and* $\beta \in (0,1]$ *such that*

$$\inf_{Q\in Y} \sigma^{-n} \int_{B_x(\sigma)} d^2(f, Q)\, d\mu_g \le K^2\sigma^{2\beta}, \forall x \in \mathbf{B}(\varrho r) \text{ and } \sigma \in (0, R), \tag{58}$$

then (there exists a representative in the L^2-equivalence class of f which we still denote by f) such that

$$d(f(x), f(y)) \leq C|x-y|^{\beta}, \ \forall x, y \in \mathbf{B}(\varrho r)$$

with C depending on K, r, R, β, ϱ and $\mathbf{B}(r)$.

PROOF. We will use C to denote a generic constant that depends only on K, r, R, β, ϱ and $\mathbf{B}$. For $x \in \mathbf{B}(\varrho r)$ and $\sigma \in (0, R)$, let $Q_{x,\sigma} \in Y$ be such that

$$\frac{1}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, Q_{x,\sigma}) \, d\mu_g = \inf_{Q \in Y} \frac{1}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, Q) \, d\mu_g.$$

For the existence of $Q_{x,\sigma}$, see Lemma 2.5.1 of [KS1]. Furthermore,

$$\begin{aligned}
\left(\frac{\sigma}{2}\right)^{-n} \int_{B_x(\sigma/2)} d^2(f, Q_{x,\sigma/2}) \, d\mu_g &\leq \left(\frac{\sigma}{2}\right)^{-n} \int_{B_x(\sigma/2)} d^2(f, Q_{x,\sigma}) \, d\mu_g \\
&\leq \left(\frac{\sigma}{2}\right)^{-n} \int_{B_x(\sigma)} d^2(f, Q_{x,\sigma}) \, d\mu_g \\
&\leq 2^n C \sigma^{2\beta}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(\frac{\sigma}{2}\right)^{-n} \int_{B_x(\sigma/2)} d^2(Q_{x,\sigma}, Q_{x,\sigma/2}) \, d\mu_g \\
&\quad \leq \left(\frac{\sigma}{2}\right)^{-n} \int_{B_x(\sigma/2)} 2d^2(f, Q_{x,\sigma}) + 2d^2(f, Q_{x,\sigma/2}) \, d\mu_g \\
&\quad \leq 2^{n+2} C \sigma^{2\beta}
\end{aligned}$$

which implies

$$d(Q_{x,\sigma}, Q_{x,\sigma/2}) \leq C\sigma^{\beta}.$$

Apply the above inequality with $\sigma = R/2^k$ and sum to obtain

$$\begin{aligned}
d(Q_{x,R/2^{\nu}}, Q_{x,R/2^{\nu'}}) &\leq \sum_{k=\nu}^{\nu'-1} d(Q_{x,R/2^k}, Q_{x,R/2^{k+1}}) \\
&\leq C \sum_{k=\nu}^{\nu'-1} \left(\frac{1}{2^{\beta}}\right)^{\nu}. \qquad (59)
\end{aligned}$$

The sum on the right hand side is a partial sum of a geometric series and hence $\{Q_{x,R/2^{\nu}}\}_{\nu=0,1,\ldots}$ is a Cauchy sequence. Since Y is complete, there exists $Q_x \in Y$ such that

$$\lim_{\nu \to \infty} d(Q_x, Q_{x,R/2^{\nu}}) = 0.$$

Again using (59), we obtain

$$\begin{aligned} d(Q_{x,R/2^\nu}, Q_x) &= \lim_{\nu' \to \infty} d(Q_{x,R/2^\nu}, Q_{x,R/2^{\nu'}}) \\ &= \lim_{\nu' \to \infty} \sum_{k=\nu}^{\nu'-1} d(Q_{x,R/2^k}, Q_{x,R/2^{k+1}}) \\ &\leq C\left(\frac{1}{2^\nu}\right)^\beta. \end{aligned}$$

Thus, using $d^2(f,Q_x) \leq 2d^2(f,Q_{x,\sigma}) + 2d^2(Q_{x,\sigma},Q_x)$, we obtain

$$\sigma^{-n} \int_{B_x(\sigma)} d^2(f,Q_x)\, d\mu_g \leq C\sigma^{2\beta} \tag{60}$$

for every $\sigma = \frac{R}{2^\nu}$, $\nu = 0, 1, \ldots$. On the other hand, for an $\sigma \in (0, R]$, there is an integer $\nu \geq 0$ such that $R/2^{\nu+1} < \sigma \leq R/2^\nu$ and we can conclude that (60) holds for all $\sigma \in (0, R]$. Let $x, y \in \mathbf{B}(\varrho r)$ with $|x - y| < R$ and let $x_{\frac{1}{2}}$ be a point in $\mathbf{B}(r)$ such that $2|x - x_{\frac{1}{2}}| = 2|y - x_{\frac{1}{2}}| = |x - y| =: \sigma$. Using the fact that $B_{x_{\frac{1}{2}}}(\frac{\sigma}{2}) \subset B_x(\sigma) \cap B_y(\sigma)$, we obtain

$$\begin{aligned} & d^2(Q_x,Q_y) \\ \leq\ & \frac{1}{Vol(B_{x_{\frac{1}{2}}}(\frac{\sigma}{2}))} \int_{B_{x_{\frac{1}{2}}}(\frac{\sigma}{2})} d^2(Q_x,Q_y)\ \ d\mu_g \\ \leq\ & C\sigma^{-n} \int_{B_{x_{\frac{1}{2}}}(\frac{\sigma}{2})} 2d^2(f,Q_x) + 2d^2(f,Q_y)\ \ d\mu_g \\ \leq\ & C\left(\sigma^{-n} \int_{B_x(\sigma)} d^2(f,Q_x)\, d\mu_g + \sigma^{-n} \int_{B_y(\sigma)} d^2(f,Q_y)\, d\mu_g\right) \\ \leq\ & C\sigma^{2\beta} = C|x-y|^{2\beta}. \end{aligned}$$

For a pair of points $x, y \in \mathbf{B}(\varrho r)$, we can choose a sequence $z_0 = x, \ldots, z_k = y$ such that $|z_i - z_{i+1}| \leq R$ and $k \leq 2\varrho r/R$. Applying the above inequality to pairs z_i, z_{i+1} and summing, we obtain

$$d^2(Q_x,Q_y) \leq \frac{2\varrho r}{R} C|x-y|^{2\beta}, \forall x, y \in \mathbf{B}(\varrho r).$$

Finally, we will show that $f(x) = Q_x$ for a.e. $x \in \mathbf{B}(\varrho r)$. This of course will complete the proof of the Lemma. It suffices to show that $f(x) = Q_x$ for a.e. x contained in an interior of a wedge. In fact, by using compact exhaustion, it suffices to show $f(x) = Q_x$ for $x \in \Omega$ where Ω is an Euclidean domain contained in the interior of every wedge of $\mathbf{B}(\varrho r)$. We first prove:

Claim. If

$$E_\epsilon = \left\{ x \in \Omega : \limsup_{\sigma \to 0} \frac{1}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, f(x))\ d\mu_g > 4\epsilon \right\}$$

then

$$\mu_g(E_\epsilon) = 0.$$

To prove this claim, we let $g : \Omega \to Y$ be a Lipschitz map which approximates f in L^2 (cf. Section 1.5 of [KS2]). Then

$$\begin{aligned}
&\frac{1}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, f(x))\ d\mu_g \\
\leq\ & \frac{2}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, g)\ d\mu_g + \frac{2}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(g, g(x))\ d\mu_g \\
&\quad + \frac{2}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(g(x), f(x))\ d\mu_g \\
\leq\ & \frac{2}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, g)\ d\mu_g + \frac{2}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(g, g(x))\ d\mu_g \\
&\quad + 2d^2(g(x), f(x)).
\end{aligned}$$

Since g is continuous, the second term on the right hand side approaches 0 as $\sigma \to 0$. Thus,

$$\limsup_{\sigma \to 0} \frac{1}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, f(x))\ d\mu_g \leq 2HL(x) + 2d^2(g(x), f(x))$$

where

$$HL(x) := \sup_{r > 0} \frac{1}{Vol(B_x(r))} \int_{B_x(r)} d^2(f, g)\ d\mu_g.$$

If

$$\limsup_{\sigma \to 0} \frac{1}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, f(x))\ d\mu_g > 4\epsilon$$

then either

$$HL(x) > \epsilon \quad \text{or} \quad d^2(g(x), f(x)) > \epsilon.$$

By the Hardy-Littlewood maximal theorem there exists a constant $c > 0$ such that

$$\mu_g\Big(\{x \in \Omega : HL(x) > \epsilon\}\Big) \leq \frac{c}{\epsilon} \int_\Omega d^2(f, g) d\mu_g$$

and the Markov inequality says

$$\mu_g\Big(\{x \in \Omega : d^2(f, g) > \epsilon\}\Big) \leq \frac{1}{\epsilon} \int_\Omega d^2(f, g) d\mu_g.$$

Hence

$$\mu_g(E_\epsilon) < \frac{c+1}{\epsilon} \int_\Omega d^2(f,g)\; d\mu_g.$$

Since g can be chosen such that the integral on the right hand side is arbitrarily small, we have proved the claim.

It follows that

$$\lim_{\sigma \to 0} \frac{1}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, f(x))\; d\mu_g = 0 \;\text{ for a.e. } x \in \Omega.$$

Furthermore,

$$\begin{aligned}
& d(f(x), Q_x) \\
& = \frac{1}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d(f(x), Q_x)\; d\mu_g \\
& \leq \frac{2}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, f(x))\; d\mu_g + \frac{2}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, Q_x)\; d\mu_g \\
& \leq \frac{2}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, f(x))\; d\mu_g + \frac{4}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, Q_{x,\sigma})\; d\mu_g \\
& \quad + \frac{4}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(Q_x, Q_{x,\sigma})\; d\mu_g \\
& \leq \frac{6}{Vol(B_x(\sigma))} \int_{B_x(\sigma)} d^2(f, f(x))\; d\mu_g + 4d^2(Q_x, Q_{x,\sigma}).
\end{aligned}$$

Hence,

$$\begin{aligned}
& d(f(x), Q_x) \\
& \leq 6 \lim_{\nu \to \infty} \frac{1}{Vol(B_x(R/2^\nu))} \int_{B_x(R/2^\nu)} d^2(f, f(x))\; d\mu_g + 4 \lim_{\nu \to \infty} d^2(Q_x, Q_{x,R/2^\nu}) \\
& = 0 \quad \text{for a.e. } x \in \Omega.
\end{aligned}$$

This completes the proof. Q.E.D.

By combining the previous Lemma with the Poincare Lemma 18, we obtain

Proposition 39 *Let* $\mathbf{B}$ *be a dimension-n, codimension-$(n-k)$ local model, g a Lipschitz Riemannian metric defined on $\mathbf{B}(r)$, (Y,d) an NPC space and $f : (\mathbf{B}(r), g) \to Y$ a finite energy map. Fix $\varrho \in (0,1)$. Suppose that there exists $R > 0$ so that for every $x \in \mathbf{B}(\varrho r), \sigma < R$,*

$$E_x(\sigma) \leq K^2 \sigma^{n-2+2\beta}. \tag{61}$$

Then

$$d(f(x), f(y)) \leq C|x-y|^\beta, \forall x, y \in \mathbf{B}(\varrho r)$$

with C depending on K, R, ϱ, $\mathbf{B}(r)$ and the ellipticity constant of g.

Theorem 40 *Let* $\mathbf{B}$ *be a local model,* g *a Lipschitz Riemannian metric defined on* $\mathbf{B}(r)$*,* (Y,d) *an NPC space,* $f : (\mathbf{B}(r), g) \to Y$ *a harmonic map and* α_x *the order of* f *at* x*. If* $\alpha_x \geq \alpha > 0$ *for all* $x \in \mathbf{B}(\varrho r)$ *where* $\varrho \in (0,1)$*, then there exists* C *depending only on the Lipchitz bound and ellipticity constant of* g*,* E^f*,* $\mathbf{B}(r)$ *and* ϱ *such that*

$$d(f(x), f(y)) \leq C|x-y|^\alpha, \quad \forall x, y \in \mathbf{B}(\varrho r).$$

Proof. The result follows immediately from inequality (45) of Proposition 32, Proposition 37 and Proposition 39. Q.E.D.

Theorem 41 *Let* $\mathbf{B}$ *be a local model,* g *a Lipschitz Riemannian metric defined on* $\mathbf{B}(r)$*,* (Y,d) *an NPC space and* $f : (\mathbf{B}(r), g) \to Y$ *a harmonic map. For* $\varrho \in (0,1)$*, there exist* C *and* $\gamma > 0$ *depending only on the Lipchitz bound and ellipticity constant of* g*,* E^f*,* $\mathbf{B}(r)$ *and* ϱ *such that*

$$d(f(x), f(y)) \leq C|x-y|^\gamma, \quad \forall x, y \in \mathbf{B}(\varrho r).$$

Proof. The result follows immediately from inequality (46) of Proposition 32, Propsition 37 and Proposition 39. Q.E.D.

By using Proposition 6 we obtain the following:

Theorem 42 *Let* $B(r)$ *be a ball or radius* r *around a point in an admissible complex* X *endowed with a Lipschitz Riemannian metric* g *and* (Y,d) *an NPC space. If* $f : (B(r), g) \to Y$ *is a harmonic map and* $\varrho \in (0,1)$*, then there exist* C *and* $\gamma > 0$ *so that*

$$d(f(x), f(y)) \leq C|x-y|^\gamma, \quad \forall x, y \in B(\varrho r).$$

Here, C *and* γ *only depend the total energy* E^f *of the map* f*,* $(B(r), g)$ *and* ϱ*.*

5 Convergence in the pull back sense

Given a metric space (Y,d) and a map $u : \mathbf{B}(r) \to (Y,d)$, we recall the following construction of [KS2]. First, we let $\Omega_0 = \mathbf{B}(r)$, $u_0 = u$ and $d_0 : \Omega_0 \times \Omega_0 \to \mathbf{R}^+ \cup \{0\}$ be the pseudodistance function $d_0(x,y) = d(u_0(x), u_0(y))$. Next, we inductively define $\Omega_{i+1} = \Omega_i \times \Omega_i \times [0,1]$ and identify Ω_i as a subset of Ω_{i+1} by the inclusion map $x \mapsto (x,x,0)$. Extend $u_i : \Omega \to (Y,d)$ to $u_{i+1} : \Omega \to (Y,d)$ by

$$u_{i+1}(x,y,\lambda) = (1-\lambda)u_i(x) + \lambda u_i(y)$$

and let

$$d_{i+1}(x,y) = d(u_{i+1}(x), u_{i+1}(y)).$$

Thus,

$$d_{i+1}((x,x,0),(y,y,0)) = d_i(x,y)$$
$$d_{i+1}((x,y,\lambda),(x,y,\mu)) = |\lambda-\mu| d_i(x,y)$$

and

$$\begin{aligned} d_{i+1}(z,(x,y,\lambda)) &\le (1-\lambda)d_{i+1}(z,(x,x,0)) \\ &+\lambda d_{i+1}(z,(y,y,0)) - \lambda(1-\lambda)d_{i+1}((x,x,0),(y,y,0)). \end{aligned} \tag{62}$$

Let $\Omega_\infty = \cup \Omega_i$ and define $u_\infty : \Omega_\infty \to (Y,d)$ by setting $u_\infty = u_i$ on Ω_i. With $d_\infty(x,y) := d(u_\infty(x), u_\infty(y))$, define (Y_∞, d_∞) as the completion of the quotient space from $(\Omega_\infty, d_\infty)$ and let $\pi : \Omega_\infty \to Y_\infty$ be the natural projection map. Equation (62) implies that the metric space (Y_∞, d_∞) is an NPC space. The unique extension of u_∞ to Y_∞ is an isometry $U : (Y_\infty, d_\infty) \to \mathcal{C}(u(\mathbf{B}(r))) \subset Y$ to the closed convex hull of the image of u. Furthermore, if $\iota : \mathbf{B}(r) = \Omega_0 \to \Omega_\infty$ is the inclusion map, then $u = U \circ \pi \circ \iota$. (cf. [KS2])

Definition 43 *Let $v_k : \mathbf{B}(r) \to (Y_k, d_k)$ be a sequence of maps to NPC spaces. We say v_k converges to v_* in the pullback sense if there exists a pseudodistance function $d_* : \Omega_\infty \times \Omega_\infty \to \mathbf{R}^+ \cup \{0\}$ with the following property. Let (Y_*, d_*) be the completed quotient space from (Ω_∞, d_*) and $\pi : \Omega_\infty \to Y_*$ the natural projection map. Furthermore, let $v_k = u$ in the above paragraph and let $d_{k,\infty} : \Omega_\infty \times \Omega_\infty \to \mathbf{R}^+ \cup \{0\}$ the corresponding pullback distance function of $v_{k,\infty}$ replacing u_∞ above. Then $d_{k,\infty}$ converges pointwise to d_* and $v_* = \pi \circ \iota$.*

Remark. If we let $v_* = u$ with u as in the paragraph preceeding the definition above and $d_{*,i}$ ($d_{*,\infty}$ resp.) the corresponding pullback distance function of $v_{*,i} = u_i$ ($v_{*,\infty} = u_\infty$ resp.), then $d_* = d_{*,\infty}$.

Definition 44 *Suppose v_k converge to v_* in the pullback sense. Let $d_{k,i}$ (resp. $d_{*,i}$) be the corresponding pullback distance function to $v_{k,i} : \Omega_i \to (Y_k, d_k)$ (resp. $v_{*,i} : \Omega_i \to (Y_*, d_*)$). We say that the convergence is locally uniform if the convergence of $d_{k,i}$ to the limit $d_{*,i}$ is uniform on each compact subset of $\Omega_i \times \Omega_i$. In this case, we also say that v_* is a locally uniform limit of v_k.*

Proposition 45 *Let $v_k : \mathbf{B}(r) \to (Y_k, d_k)$ be a sequence of maps to NPC spaces for which there is uniform modulus of continuity control, i.e. assume for each $x \in \mathbf{B}(r)$ and $R > 0$ there is a positive function $\omega(x,R)$ which is monotone in R, satisfying*

$$\lim_{R \to 0} \omega(x,R) = 0,$$

and so that for each $k \in \mathbf{Z}$

$$\max_{y \in B(x,R)} d(v_k(x), v_k(y)) \le \omega(x,R).$$

Then there is an NPC space (Y_*, d_*) and a subsequence $v_{k'}$ of the v_k which converges locally uniformly in the pullback sense to a limit map $v_* : \mathbf{B}(r) \to (Y_*, d_*)$, and v_* satisfies the same modulus of continuity estimates. Here, (Y_*, d_*) is the completed quotient of $(\Omega_\infty, d_{*,\infty})$ where $d_{*,\infty} = \lim_{k' \to \infty} d_{k',\infty}$.

PROOF. The proposition follows from the argument of the proof of Lemma 3.1 and Proposition 3.7 in [KS2] since the fact that $\mathbf{B}(r)$ is not a Riemannian domain plays no consequence in the argument. Q.E.D.

6 The tangent map

Let $\mathbf{B}$ be a dimension-n, codimension-ν local model and g a normalized (i.e. $g_{ij}(0) = \delta_{ij}$) Lipschitz metric on $\mathbf{B}(1)$. Given $r \in (0,1)$, $f : \mathbf{B}(r) \to (Y, d)$ and $\lambda > 0$, define the λ-*blow up map* $f_\lambda : \mathbf{B}(\frac{r}{\lambda}) \to (Y, d_\lambda)$ by setting:

$$\begin{aligned} g_\lambda(x) &= g(\lambda x) \\ \mu_\lambda &= (\lambda^{1-n} I(\lambda))^{1/2} \\ d_\lambda(p,q) &= \mu_\lambda^{-1} d(p,q) \\ f_\lambda(x) &= f(\lambda x). \end{aligned}$$

Definition 46 *If there exist $\lambda_k \to 0$ and an NPC space (Y_*, d_*) so that f_{λ_k} converges locally uniformly in the pullback sense to $f_* : \mathbf{B} \to (Y_*, d_*)$, then f_* is called a tangent map of f.*

Lemma 47 *A harmonic map $f : (\mathbf{B}, g) \to (Y, d)$ has a non-constant tangent map f_* which satisfies*

$$d(f_*(x), f_*(y)) \leq C'|x - y|^\gamma$$

where C' and γ are only depending on the Lipschitz bound of g and E^f.

PROOF. By change of variables

$$\int_{\mathbf{B}(\sigma)} |\nabla f_\lambda|^2_{g_\lambda} d\mu_{g_\lambda} = \mu_\lambda^{-2} \lambda^{2-n} \int_{\mathbf{B}(\lambda\sigma)} |\nabla f|^2_g d\mu_g$$

and

$$\int_{\partial \mathbf{B}(\sigma)} d^2_\lambda(f_\lambda, f_\lambda(0)) d\Sigma_{g_\lambda} = \mu_\lambda^{-2} \lambda^{1-n} \int_{\partial \mathbf{B}(\lambda\sigma)} d^2(f, f(0)) d\Sigma_g.$$

Thus, the definition of μ_λ implies

$$\int_{\partial \mathbf{B}(1)} d^2_\lambda(f_\lambda, f_\lambda(0)) d\Sigma_{g_\lambda} = 1$$

and Corollary 31 implies

$$\lim_{\lambda\to 0}\frac{\int_{\mathbf{B}(1)}|\nabla f_\lambda|^2_{g_\lambda}d\mu_{g_\lambda}}{\int_{\partial\mathbf{B}(1)}d^2_\lambda(f_\lambda,f_\lambda(0))d\Sigma_{g_\lambda}}=\lim_{\lambda\to 0}\frac{\lambda E(\lambda)}{I(\lambda)}=\alpha.$$

Consequently, by choosing λ sufficiently small, we have

$$\int_{\mathbf{B}(1)}|\nabla f_\lambda|^2 d\mu_{g_\lambda}\leq 2\alpha. \tag{63}$$

Since g_{ij} is Lipschitz, we have

$$|g_{ij}(x)-\delta_{ij}|\leq c|x| \text{ and } |g^{ij}(x)-\delta_{ij}|\leq c|x|. \tag{64}$$

Hence,

$$|(g_\lambda)_{ij}(x)-\delta_{ij}|\leq c\lambda|x| \text{ and } |(g_\lambda)^{ij}-\delta_{ij}|\leq c\lambda|x|. \tag{65}$$

Therefore, there exists a uniform Lipschitz bound of the family of metrics $\{g_\lambda\}$ independent of λ. This implies

$$d_\lambda(f_\lambda(x),f_\lambda(y))\leq C|x-y|^\gamma \ \forall x,y\in\mathbf{B}(r),$$

where C and γ are independent of λ by Theorem 41. Thus, we have uniform modulus of continuity control for the sequence f_λ, then by Proposition 45, there exists a sequence $\lambda_k\to 0$ and an NPC space (Y_*,d_*) so that f_{λ_k} converges locally uniformly in the pullback sense to a limit map $f_*:\mathbf{B}(1)\to(Y_*,d_*)$. The fact that f_* is non-constant follows immediately from the proof of Proposition 3.3 of [GS]. Q.E.D.

Lemma 48 *Let g be a normalized metric on $\mathbf{B}(r)$, $f:(\mathbf{B}(r),g)\to(Y,d)$ a harmonic map and $f_\lambda:\mathbf{B}(\frac{r}{\lambda})\to(Y,d_\lambda)$ be the λ-blow up map. Let $h_\lambda:\mathbf{B}(1)\to(Y,d_\lambda)$ be a map which is harmonic with respect to the Euclidean domain metric and with the boundary condition $h_\lambda|_{\partial\mathbf{B}(1)}=f_\lambda|_{\partial\mathbf{B}(1)}$. Then*

$$(1-c\lambda)\ {}^{\delta}E^{f_\lambda}\leq\ {}^{g_\lambda}E^{f_\lambda}\leq(1+c\lambda)\ {}^{\delta}E^{f_\lambda} \tag{66}$$

and

$$(1-c\lambda)\ {}^{\delta}E^{h_\lambda}\leq\ {}^{g_\lambda}E^{h_\lambda}\leq(1+c\lambda)\ {}^{\delta}E^{h_\lambda}. \tag{67}$$

Proof. By inequality (65), we have

$$(1-c\lambda)\sum_{i,j=1}^n\delta^{ij}\frac{\partial f_\lambda}{\partial x_i}\cdot\frac{\partial f_\lambda}{\partial x_j}\leq\sum_{i,j=1}^n(g_\lambda)^{ij}\frac{\partial f_\lambda}{\partial x_i}\cdot\frac{\partial f_\lambda}{\partial x_j}\leq(1+c\lambda)\sum_{i,j=1}^n\delta^{ij}\frac{\partial f_\lambda}{\partial x_i}\cdot\frac{\partial f_\lambda}{\partial x_j} \tag{68}$$

and

$$(1-c\lambda)\sum_{i,j=1}^{n}\delta^{ij}\frac{\partial h_\lambda}{\partial x_i}\cdot\frac{\partial h_\lambda}{\partial x_j}\le\sum_{i,j=1}^{n}(g_\lambda)^{ij}\frac{\partial h_\lambda}{\partial x_i}\cdot\frac{\partial h_\lambda}{\partial x_j}\le(1+c\lambda)\sum_{i,j=1}^{n}\delta^{ij}\frac{\partial h_\lambda}{\partial x_i}\cdot\frac{\partial h_\lambda}{\partial x_j}. \tag{69}$$

The assertion follows immediately. Q.E.D.

In particular, Lemma 48 and (63) imply that

$$^{\delta}E^{h_\lambda}\le\ ^{\delta}E^{f_\lambda}\le\frac{1}{1-c\lambda}\ ^{g_\lambda}E^{f_\lambda}\le\frac{2\alpha}{1-c\lambda}. \tag{70}$$

Thus, Proposition 45 and Theorem 41 imply that there exists a subsequence of λ_k (which we will still denote λ_k by an abuse of notation) and an NPC space $(\bar{Y}_*,\bar{d}_*)$ so that h_{λ_k} converge locally uniformly in the pullback sense to $h_*:\mathbf{B}(1)\to(\bar{Y}_*,\bar{d}_*)$. Set $h_k:=h_{\lambda_k}$, $f_k:=f_{\lambda_k}$ and $g_k=g_{\lambda_k}$. Furthermore, let $d_k(x,y)=d_{\lambda_k}(f_k(x),f_k(y))$ and $\bar{d}_k(x,y)=d_{\lambda_k}(h_k(x),h_k(y))$. Then in any compactly contained subset of $\mathbf{B}(1)\times\mathbf{B}(1)$, d_k, $\bar{d}_k$ converge uniformly to (the restriction to $\mathbf{B}(1)=\Omega_0$ of) d_*, $\bar{d}_*$ respectively.

Proposition 49 *Under the notation above, the pseudodistance functions d_* and $\bar{d}_*$ above are equal. Consequently, $f_*=h_*$ and h_k (which are harmonic maps with respect to the Euclidean metric) converge locally uniformly in the pullback sense to f_*.*

PROOF. By the repeated use of the triangle inequality,

$$|d_k(x,y)-\bar{d}_k(x,y)|\le d_{\lambda_k}(f_k(x),h_k(x))+d_{\lambda_k}(f_k(y),h_k(y)).$$

Therefore, for any $r<1$, the Lebesgue dominated convergence theorem and the Poincaré inequality (cf. Theorem 16) imply

$$\begin{aligned}
&\int_{\mathbf{B}(r)}\int_{\mathbf{B}(r)}|d_*(x,y)-\bar{d}_*(x,y)|^2d\mu(x)d\mu(y)\\
&=\lim_{k\to0}\int_{\mathbf{B}(r)}\int_{\mathbf{B}(r)}|d_k(x,y)-\bar{d}_k(x,y)|^2d\mu(x)d\mu(y)\\
&\le 4\mathrm{vol}(\mathbf{B}(r))\lim_{k\to0}\int_{\mathbf{B}(r)}d^2_{\lambda_k}(f_k(x),h_k(x))d\mu(x)\\
&\le 4C\lim_{k\to0}\int_{\mathbf{B}(r)}|\nabla d^2_{\lambda_k}(f_k(x),h_k(x))|d\mu(x).
\end{aligned} \tag{71}$$

Equations (66) and (67) imply

$$^{\delta}E^{f_\lambda}\le\frac{1}{1-c\lambda}\ ^{g_\lambda}E^{f_\lambda}\le\frac{1}{1-c\lambda}\ ^{g_\lambda}E^{h_\lambda}\le\frac{1+c\lambda}{1-c\lambda}\ ^{\delta}E^{h_\lambda}.$$

Therefore, if we let $w = \frac{1}{2}f_\lambda + \frac{1}{2}h_\lambda$,

$$\begin{aligned} 2\ {}^\delta E^w &\leq {}^\delta E^{f_\lambda} + {}^\delta E^{h_\lambda} - \frac{1}{2}\int_{\mathbf{B}(r)} |\nabla d_\lambda^2(f_\lambda, h_\lambda)| d\mu \\ &= 2\ {}^\delta E^{h_\lambda} + O(\lambda) - \frac{1}{2}\int_{\mathbf{B}(r)} |\nabla d_\lambda^2(f_\lambda, h_\lambda)| d\mu \end{aligned}$$

by equation (2.2iv) of [KS1]. Since ${}^\delta E^{h_\lambda} \leq {}^\delta E^w$, this in turn implies

$$\int_{\mathbf{B}(r)} |\nabla d_\lambda^2(f_\lambda, h_\lambda)| d\mu \to 0$$

as $\lambda \to 0$. This, combined with equation (71) the continuity of d_* and $\bar{d}_*$, implies that $d_*(x,y) = \bar{d}_*(x,y)$ which in turn implies that $(Y_*, d_*) = (\bar{Y}_*, \bar{d}_*)$ and $h_* = f_*$. Q.E.D.

Lemma 50 *Assuming that the directional energies of h_k converge to those of f_*, the tangent map $f_* : \mathbf{B}(1) \to Y_*$ is homogenous of order α where α is the order of f at 0, i.e.*

$$d_*(f_*(x), f_*(0)) = |x|^\alpha d_*(f_*(\frac{x}{|x|}), f_*(0))$$

and the image of $t \mapsto f_*(tx)$, $0 \leq t \leq 1$, is a geodesic.

Proof. For notational simplicity in this proof we will let $E = {}^\delta E$ and $I = {}^\delta I$. Using (8) and (13) with f replaced by h_k, noting that the remainder in (8) is 0 because the domain is Euclidean, and using the convergence of h_k and its directional energies to f_* and its directional energies, we have

$$\left(E^{f_*}(\sigma)\right)' = 2\int_{\partial\mathbf{B}(\sigma)} \left|\frac{\partial f_*}{\partial r}\right|^2 d\Sigma$$

and

$$2E^{f_*}(\sigma) \leq \int_{\partial\mathbf{B}(\sigma)} d(f_*, f_*(0))\frac{\partial}{\partial r} d(f_*, f_*(0)) d\Sigma.$$

We next claim

$$\begin{aligned} &\left(\log\left(\frac{\sigma E^{f_*}(\sigma)}{I^{f_*}(\sigma)}\right)\right)' \\ \geq\ & \frac{2}{E^{f_*}(\sigma) I^{f_*}(\sigma)}\left[\left(\int_{\mathbf{B}(\sigma)} d^2(f_*, f_*(0)) d\Sigma\right)\left(\int_{\mathbf{B}(\sigma)} \left|\frac{\partial f_*}{\partial r}\right|^2 d\Sigma\right)\right. \\ & \left. - \int_{\partial\mathbf{B}(\sigma)} \left(d(f_*, f_*(0))\frac{\partial}{\partial r} d(f_*, f_*(0)) d\Sigma\right)^2\right] \geq 0. \end{aligned} \tag{72}$$

This follows from (11), (16) and (21) applied to the harmonic map h_k (without the error term due to the fact that h_k is harmonic for the Euclidean domain metric) and the assumption that the directional energies of h_k converge to those of f_*. On the other hand, our assumption on the convergence of directional energies implies that

$$\begin{aligned}
\frac{\sigma E^{f_*}(\sigma)}{I^{f_*}(\sigma)} &= \lim_{k\to\infty} \frac{\sigma E^{f_k}_{g_k}(\sigma)}{I^{f_k}_{g_k}(\sigma)} \\
&= \lim_{k\to\infty} \frac{\sigma \mu_{\lambda_k}^{-2}\lambda_k^{2-n}\ {}^gE^f(\lambda_k\sigma)}{\mu_{\lambda_k}^{-2}\lambda_k^{1-n}\ {}^gI^f(\lambda_k\sigma)} \\
&= \lim_{k\to\infty} \frac{\sigma\lambda_k\ {}^gE^f(\lambda_k\sigma)}{{}^gI^f(\lambda_k\sigma)} \\
&= \alpha.
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &= \left(\log\left(\frac{\sigma E^{f_*}(\sigma)}{I^{f_*}(\sigma)}\right)\right)' \\
&= \frac{2}{E^{f_*}(\sigma)I^{f_*}(\sigma)}\left[\left(\int_{\mathbf{B}(\sigma)} d^2(f_*, f_*(0))d\Sigma\right)\left(\int_{\mathbf{B}(\sigma)}\left|\frac{\partial f_*}{\partial r}\right|^2 d\Sigma\right)\right. \\
&\quad \left. - \int_{\partial\mathbf{B}(\sigma)}\left(d(f_*, f_*(0))\frac{\partial}{\partial r}d(f_*, f_*(0))d\Sigma\right)^2\right].
\end{aligned}$$

Hence,

$$\frac{\partial}{\partial r}d(f_*, f_*(0)) = \left|\frac{\partial f_*}{\partial r}\right| \quad \text{a.e.}$$

and

$$2\int_{\mathbf{B}(\sigma)} |\nabla f_*|^2 d\mu = \int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}(d^2(f_*, f_*(0)))d\Sigma.$$

We can now follow the proof of Proposition 3.1 [GS] to show the homogeneity of f_*. Q.E.D.

Lemma 51 *Let $f_* : \mathbf{B}(1) \to (Y_*, d_*)$ be a homogeneous map of order α. (See definition of homogeneity in Lemma 50.) Then there exist a metric space $(C, \hat{d})$ and a map $\hat{f}_* : \mathbf{B}(1) \to C$ so that the energy density of f_* is equal to that of $\hat{f}_*$ and for every $x, y \in \mathbf{B}(r)$ and $x' = tx$, $y' = ty$, we have*

$$\hat{d}(\hat{f}_*(x'), \hat{f}_*(y')) = t^\alpha \hat{d}(\hat{f}_*(x), \hat{f}_*(y)). \tag{73}$$

PROOF. Let C be the disjoint union of geodesics from $f_*(0)$ to $f_*(x)$ for each $x \in \partial \mathbf{B}(1)$ with $f_*(0)$ identified. We define a distance function $\hat{d}$ on C in the following way. Let $P, Q \in C$ and suppose that P (resp. Q) is the point on the geodesic γ (resp. σ) from $f_*(0)$ to $f_*(x)$ (resp. $f_*(y)$) at a distance r (resp. s) from $f_*(0)$, where $x, y \in \partial \mathbf{B}(1)$. We first define the angle θ between γ and σ by

$$\cos\theta = \frac{d_*^2(f_*(0), f_*(x)) + d_*^2(f_*(0), f_*(y)) - d_*^2(f_*(x), f_*(y))}{2d_*(f_*(0), f_*(x))d_*(f_*(0), f_*(y))}$$

and set

$$\hat{d}^2(P,Q) = r^2 + s^2 - 2rs\cos\theta,$$
$$\hat{f}_*(x) = f_*(x).$$

By definition of $\hat{d}$, we see that (73) holds. Therefore,

$$\hat{d}(\hat{f}_*(x), \hat{f}_*(y)) = d_*(f_*(x), f_*(y))$$

whenever x, y lie on the same geodesic from 0 or whenever $x, y \in \partial \mathbf{B}(1)$. Therefore, for any $x \in \partial \mathbf{B}(r)$ and any vector V normal to $\partial \mathbf{B}(r)$,

$$|f_*(V)|^2(x) = |\hat{f}_*(V)|^2(x).$$

Furthermore, the same holds for any $x \in \partial \mathbf{B}(1)$ and V tangential to $\partial \mathbf{B}(1)$. For a.e $x = (r, \theta) \in \mathbf{B}(r)$ and a vector V tangential to $\partial \mathbf{B}(r)$, [KS1] Lemma 1.9.4 implies that

$$\begin{aligned}
|\hat{f}_*(V)|^2(r,\theta) &= \lim_{\epsilon\to 0} \frac{\hat{d}^2(\hat{f}_*(r,\theta), \hat{f}_*(r,\theta+\epsilon V))}{\epsilon^2} \\
&= r^{2\alpha} \lim_{\epsilon\to 0} \frac{\hat{d}^2(\hat{f}_*(1,\theta), \hat{f}_*(1,\theta+\epsilon V))}{\epsilon^2} \\
&= r^{2\alpha} \lim_{\epsilon\to 0} \frac{d_*^2(f_*(1,\theta), f_*(1,\theta+\epsilon V))}{\epsilon^2} \\
&= r^{2\alpha} |f_*(V)|^2(1,\theta) \\
&= |f_*(V)|^2(r,\theta)
\end{aligned}$$

where (r, θ) is the polar coordinates of x and $\epsilon \mapsto \theta + \epsilon V$ is the flow along $\partial \mathbf{B}(r)$ defined by V. Q.E.D.

7 Harmonic maps from a flat domain

Let $\mathbf{B}$ be a dimension-n codimension-ν local model with wedges $W_k \subset \mathbf{R}^n$, $k = 1, ..., N$. Recall that the coordinates $(x^1, ..., x^n)$ of $\mathbf{R}^n$ are arranged so that D is given by $x^{n-\nu+1} = ... = x^n = 0$. In this section, we show that harmonic maps $h : (\mathbf{B}(r), \delta) \to (Y, d)$ are Lipschitz in the direction parallel to $D(r) = D \cap \mathbf{B}(r)$.

Lemma 52 *Let $\phi : \mathbf{B}(1) \to \mathbf{R}$ be a non-negative L^2 function. Suppose that for $x \in \mathbf{B}(r)$, there exists $\hat{C} > 0$ so that*

$$\frac{1}{\sigma^n} \int_{B_x(\sigma)} \phi \, d\mu \leq \frac{\hat{C}}{\rho^n} \int_{B_x(\rho)} \phi \, d\mu, \;\; 0 < \sigma \leq \rho \leq \min\{R(x), 1 - r\}.$$

There exist $C > 0$ and $R \in (0, 1 - r)$ depending only on r and $\mathbf{B}(r)$ such that

$$\frac{1}{\sigma^n} \int_{B_x(\sigma)} \phi \, d\mu \leq C \int_{B_x(R)} \phi \, d\mu, \;\; \forall x \in \mathbf{B}(r), \sigma \in (0, R].$$

Proof. The result follows from the same argument contained in the proofs of Corollary 36 and Proposition 37 (with ϕ replacing $|\nabla f|^2$, $\lambda = 1$ and $\beta = 1$). Q.E.D.

Lemma 53 *Let $h : (\mathbf{B}(1), \delta) \to (Y, d)$ be a harmonic map. Let V be a unit vector parallel to $D(r)$ and let $H(x) = h(x + \epsilon V)$ for $0 < \epsilon << 1$. Then*

$$0 \leq - \int_{\mathbf{B}(1)} \nabla \eta \cdot \nabla d^2(h, H) d\mu \tag{74}$$

for $\eta \in C_c^\infty(\mathbf{B}(1 - \epsilon))$.

Proof. Define a map $h_\eta : \mathbf{B}(1 - \epsilon) \to \mathbf{R}$ by setting

$$h_\eta(x) = (1 - \eta(x))h(x) + \eta(x)H(x).$$

Here, $(1 - t)P + tQ$ for $P, Q \in Y$ denotes the point on the unique geodesic between P and Q at a distance $td(P, Q)$ from P and $(1 - t)d(P, Q)$ from Q. Since $\text{spt}(\eta) \subset \mathbf{B}(1 - \epsilon)$, we see that

$$h_\eta|_{\partial \mathbf{B}(1-\epsilon)} = h|_{\partial \mathbf{B}(1-\epsilon)}$$

and

$$h_{1-\eta}|_{\partial \mathbf{B}(1-\epsilon)} = H|_{\partial \mathbf{B}(1-\epsilon)}.$$

By following the proofs of Lemma 2.4.1 and 2.4.2 of [KS1], we see that $h_\eta, h_{1-\eta} \in W^{1,2}(\mathbf{B}(r))$, and on each wedge W_k, $k = 1, ..., N$, we have

$$\begin{aligned}
&\int_{\mathbf{B}(1-\epsilon)\cap W_k} |\nabla h_\eta|^2 d\mu + \int_{\mathbf{B}(1-\epsilon)\cap W_k} |\nabla h_{1-\eta}|^2 d\mu \\
&\leq \int_{\mathbf{B}(1-\epsilon)\cap W_k} |\nabla h|^2 d\mu + \int_{\mathbf{B}(1-\epsilon)\cap W_k} |\nabla H|^2 d\mu \\
&\quad -2 \int_{\mathbf{B}(1-\epsilon)\cap W_k} \nabla \eta \cdot \nabla d^2(h, H) d\mu + \int_{\mathbf{B}(1-\epsilon)\cap W_k} Q(\eta, \nabla \eta) d\mu
\end{aligned}$$

where $Q(\eta, \nabla\eta)$ consists of integrable terms which are quadratic in η and $\nabla\eta$. Taking the sum over $k = 1, ..., l$ and noting that the harmonicity of h and H implies

$$\int_{\mathbf{B}(1-\epsilon)} |\nabla h|^2 d\mu \leq \int_{\mathbf{B}(1-\epsilon)} |\nabla h_\eta|^2 d\mu$$

and

$$\int_{\mathbf{B}(1-\epsilon)} |\nabla H|^2 d\mu \leq \int_{\mathbf{B}(1-\epsilon)} |\nabla h_{1-\eta}|^2 d\mu,$$

we deduce

$$0 \leq -2 \int_{\mathbf{B}(1-\epsilon)} \nabla\eta \cdot \nabla d^2(h, H) d\mu + \int_{\mathbf{B}(1-\epsilon)} Q(\eta, \nabla\eta) d\mu.$$

By replacing η by $t\eta$, dividing by t and letting $t \to 0$, we obtain (74). Q.E.D.

Lemma 54 *Let $h : (\mathbf{B}(1), \delta) \to (Y, d)$ be a harmonic map and let V be a unit vector parallel to $D(1)$. For $r \in (0, 1)$ and $x \in \mathbf{B}(r)$, there exists a constant C depending only on r and $\mathbf{B}(r)$ so that*

$$|h_*(V)|^2(x) \leq C\ {}^\delta E^h.$$

PROOF. Let H be as in the proof of Lemma 53. For $x \in \mathbf{B}(1)$ and $\sigma \in (0, R(x))$, let η approximate the characteristic function of $B_x(\sigma)$ in (74) to obtain

$$\int_{\partial B_x(\sigma)} \frac{\partial}{\partial r} d^2(h, H) d\Sigma \geq 0.$$

Let

$$J_x(\sigma) = \int_{\partial B_x(\sigma)} d^2(h, H) d\Sigma \quad \text{and} \quad K_x(\sigma) = \int_{B_x(\sigma)} d^2(h, H) d\mu.$$

Then

$$J_x'(\sigma) = \int_{\partial B_x(\sigma)} \frac{\partial}{\partial r} d^2(h, H) d\Sigma + \frac{n-1}{\sigma} J_x(\sigma) \geq \frac{n-1}{\sigma} J_x(\sigma), \ \forall \sigma \in (0, R(x)).$$

This implies that

$$\left(\frac{J_x(\sigma)}{\sigma^{n-1}} \right)' \geq 0,$$

and hence

$$J_x(\tau) \leq \frac{J_x(\sigma)}{\sigma^{n-1}} \tau^{n-1}, \quad 0 < \tau \leq \sigma \leq R(x).$$

Now integrate the above inequality from $\tau = 0$ to $\tau = \sigma$ to obtain

$$K_x(\sigma) \leq \frac{\sigma J_x(\sigma)}{n} = \frac{\sigma K_x'(\sigma)}{n}.$$

Thus,

$$\left(\frac{K_x(\sigma)}{\sigma^n}\right)' = \frac{1}{\sigma^n}\left(K_x'(\sigma) - \frac{nK_x(\sigma)}{\sigma}\right) \geq 0.$$

This implies $\sigma \mapsto \frac{K_x(\sigma)}{\sigma^n}$ is non-decreasing for $\sigma \in (0, R(x))$ and hence

$$\frac{K_x(\sigma)}{\sigma^n} \leq \frac{K_x(\rho)}{\rho^n}, \; 0 < \sigma \leq \rho \leq R(x).$$

By Lemma 52, there exists $C > 0$ and $R > 0$ such that

$$\frac{K_x(\sigma)}{\sigma^n} \leq CK_x(R), \; \forall x \in \mathbf{B}(r).$$

Fix $x \in \mathbf{B}(r)$ and let $\sigma \to 0$ to obtain

$$d^2(h(x), H(x)) \leq C \int_{B_x(R)} d^2(h, H) \; d\mu.$$

Divide by ϵ^2 and let $\epsilon \to 0$ to obtain

$$|h_*(V)|^2(x) \leq C \int_{B_x(R)} |h_*(V)|^2 d\mu \leq C \; {}^{\delta}E^h.$$

Q.E.D.

Lemma 55 *Let $h : (\mathbf{B}(1), \delta) \to (Y, d)$ be a harmonic map and $r \in (0, 1)$. If x, y are a pair of points in a wedge of $\mathbf{B}(r)$ equidistant to $D(r)$, then*

$$d(h(x), h(y)) \leq L|x - y|$$

for some constant L depending only on ${}^{\delta}E^h$, r and $\mathbf{B}(1)$.

PROOF. Let $\gamma : [0, 1] \to (Y, d)$ be a constant speed parameterization of the line between x and y. Then by Lemma 54,

$$d(h(x), h(y)) \leq \int_0^1 |h_*(\gamma'(t))| dt \leq \sqrt{C' \; {}^{\delta}E^h}|x - y|.$$

Q.E.D.

8 Lipschitz regularity

8.1 At a regular point

In this subsection, we use the results of Section 7 to give a new proof of the Lipschitz regularity of Korevaar-Schoen [KS1] and generalize their result for Lipschitz domain metrics. Recall that a dimension-n, codimension-0 local model is $\mathbf{B} = \mathbf{R}^n$.

Lemma 56 *Let* $\mathbf{B}$ *be a dimension-n, codimension-0 local model, g a normalized Lipschitz metric defined on $\mathbf{B}(r)$, (Y, d) an NPC space and $f : (\mathbf{B}(r), g) \to (Y, d)$ a harmonic map. Then the order α of f at 0 is ≥ 1.*

Proof. By Proposition 49, a tangent map f_* of f is a locally uniform limit of a sequence of harmonic maps h_k from a Euclidean unit ball $\mathbf{B}(1)$. The regularity result of [GS] implies that h_k is locally Lipschitz with the local Lipschitz bound depending on ${}^{\delta}E^{h_k}$ and the distance to $\partial\mathbf{B}(1)$. Hence, so is f_*. By [KS2] Theorem 3.11, the energy densities of h_k converge to those of f_*. By Lemma 50, f_* is a homogeneous map of order α. The homogeneity and the Lipschitz continuity of f_* implies $\alpha \geq 1$. Q.E.D.

Theorem 57 *Let* $\mathbf{B}$ *be a dimension-n, codimension-0 local model, g a Lipschitz Riemannian metric on $\mathbf{B}(r)$, (Y, d) an NPC space and $f : (\mathbf{B}(r), g) \to (Y, d)$ a harmonic map. Then f is Lipschitz continuous in $\mathbf{B}(\varrho r)$ with the Lipschitz constant depending on $\varrho \in (0, 1)$, $(\mathbf{B}(r), g)$ and the total energy E^f of f.*

Proof. For each $x \in \mathbf{B}(1)$, the normalized map $f \circ L_x$ (cf. Proposition 6) has order ≥ 1 at 0 by Lemma 56. Thus, the order of f at x is ≥ 1. The result now follows from Theorem 40. Q.E.D.

8.2 At a codimension-1 singular point

Throughout this subsection $\mathbf{B}$ is a dimension-n, codimension-1 local model with wedges half spaces given by $x^n \geq 0$ and lower dimensional stratum D a hyperplane given by the equation $x^n = 0$. We first prove some properties of harmonic maps from this local model equipped with the Euclidean metric δ.

Lemma 58 *Let $h : (\mathbf{B}(1), \delta) \to (Y, d)$ be a harmonic map. For every $\beta, r \in (0, 1)$, there exists B depending only on β, r, $\mathbf{B}(1)$ and ${}^{\delta}E^h$ so that*

$$d(h(x), h(y)) \leq B|x - y|^{\beta}$$

for every $x, y \in \mathbf{B}(r)$.

Proof. By Lemma 55, h is Lipschitz when restricted to $D(t_0)$, $t_0 = \frac{r+1}{2}$. Thus Hölder regularity of h restricted to a wedge W with any Hölder exponent $\beta \in (0,1)$ follows from the boundary regularity result of Serbinowski [Se] where the Hölder constant B is only depending on the choice of β, r and the total energy of the map h. Q.E.D.

The next lemma gives an estimate of the energy decay of harmonic maps along an ϵ-neighborhood.

Lemma 59 *Let h_k be defined as in Section 6 (see the paragraph preceeding Proposition 49) and fix $R \in (0,1)$. Set $D_\epsilon(r)$ to be the ϵ-neighborhood of $D(r)$ in $\mathbf{B}(r)$, i.e.*

$$D_\epsilon(r) = \cup\{x = (x_1, ..., x_n) \in W : x_n \leq \epsilon\} \cap \mathbf{B}(r)$$

where the union is over the wedges W containing D. Then for any $r \in (0, R)$, there exist constants $C, \kappa > 0$, k_0 sufficiently large and $\epsilon_0 > 0$ sufficiently small (depending only on R) so that

$${}^{\delta}E^{h_k}[D_\epsilon(r)] \leq C\epsilon^\kappa, \ \forall k > k_0, \epsilon < \epsilon_0$$

Proof. Let $B_x(r)$ be a ball of radius r centered at x. We will use the notation,

$$E_x^h(r) = \ {}^{\delta}E_x^h(r) = \int_{B_x(r)} |\nabla h|^2 d\mu$$

and

$$I_x^h(r) = \ {}^{\delta}I_x^h(r) = \int_{\partial B_x(r)} d^2(h, h(x)) d\Sigma$$

for any map $h : \mathbf{B}(r) \to (Y, d)$. Let $r_0 := 1 - R$. By Lemma 47, f_* is a non-constant, continuous map and hence there exists $c_1 > 0$ so that

$$I_x^{f_*}(r_0) \geq 2c_1, \forall x \in D(R).$$

Thus, by the local uniform convergence, there exists k_0 so that

$$I_x^{h_k}(r_0) \geq c_1, \ \forall x \in D(R), k > k_0.$$

We may assume we have chosen k sufficiently large so that $\lambda_{k_0} \in (0, \frac{1}{Nc})$. By (70),

$$E_x^{h_k}(r_0) \leq E^{h_k}(1) \leq E^{f_k}(1) \leq \frac{2\alpha}{1 - c\lambda_{k_0}}, \ \forall x \in D(R), k > k_0.$$

Thus,

$$\frac{r_0 E_x^{h_k}(r_0)}{I_x^{h_k}(r_0)} \leq \frac{2r_0\alpha}{(1 - c\lambda_{k_0})c_1} =: c_2, \ \forall x \in D(R), k > k_0.$$

By Corollary 31,

$$\frac{\epsilon E_x^{h_k}(\epsilon)}{I_x^{h_k}(\epsilon)} \leq c_3, \ \forall x \in D(R), k > k_0, \epsilon \leq r_0$$

with c_3 depending on c_2. By Lemma 58,

$$E_x^{h_k}(\epsilon) \leq \frac{c_3 I_x^{h_k}(\epsilon)}{\epsilon} \leq \frac{c_3 B^2 \epsilon^{2\beta+n-1}}{\epsilon} = c_3 B^2 \epsilon^{2\beta+n-2}, \ \forall x \in D(R), k > k_0.$$

Here, we have choosen $\beta \in (1/2, 1)$. Since $D_\epsilon(r)$ can be covered by $\left(\frac{c_4}{\epsilon}\right)^{n-1}$ number of (2ϵ)-balls centered at points in $D(r)$ where c_4 is independent of ϵ,

$$\begin{aligned} E^{h_k}[D_\epsilon(r)] &\leq c_3 B^2 (2\epsilon)^{2\beta+n-2} \times \left(\frac{c_4}{\epsilon}\right)^{n-1} \\ &=: c_5 \epsilon^{2\beta-1}. \end{aligned}$$

The result follows from the fact that the choice of β implies $2\beta - 1 > 0$. Q.E.D.

Lemma 60 *Let h_k, f_k and $h_* = f_*$ be defined as in Section 6. For $r \in (0,1)$,*

$$\lim_{k\to 0} {}^{\delta}E^{h_k}(r) = {}^{\delta}E^{h_*}(r) \tag{75}$$

and

$$\lim_{k\to\infty} {}^{g_k}E^{f_k}(r) = {}^{\delta}E^{f_*}(r), \tag{76}$$

and the directional energies of f_k, and h_k converge to that of f_*.

Proof. Again in this proof we will denote ${}^{\delta}E = E$. By the regularity result of harmonic maps from smooth domains ([KS1] Theorem 2.4.6) or Theorem 57, h_k is uniformly Lipschitz in $\mathbf{B}(\frac{1+r}{2}) - D_{\frac{\epsilon}{2}}(\frac{1+r}{2})$ for $r \in (0,1)$. First, we notice that

$$\lim_{k\to\infty} E^{h_k}[\mathbf{B}(r) - D_\epsilon(r)] = E^{f_*}[\mathbf{B}(r) - D_\epsilon(r)].$$

This follows from [KS1] Theorem 3.11. By Lemma 59,

$$E^{h_k}[D_\epsilon(r)] \leq C\epsilon^\kappa$$

for any ϵ sufficiently small. Thus,

$$\begin{aligned} \limsup_{k\to\infty} E^{h_k}(r) - C\epsilon^\kappa &\leq \limsup_{k\to\infty} E^{h_*}[\mathbf{B}(r) - D_\epsilon(r)] \\ &= E^{f_*}[\mathbf{B}(r) - D_\epsilon(r)] \\ &\leq E^{f_*}(r). \end{aligned}$$

By lower semicontinuity of energy,

$$E^{f_*}(r) \le \liminf_{k\to\infty} E^{h_k}(r) \le \limsup_{k\to\infty} E^{h_k}(r) \le E^{f_*}(r) + C\epsilon^{\kappa}.$$

Since $\epsilon > 0$ can be made arbitrarily small, this proves (75). To prove (76), we see that

$$\begin{aligned} E^{f_*}(r) &\le \liminf_{k\to\infty}(1 - c\lambda_k)\ E^{f_k}(r) \\ &\le \liminf_{k\to\infty}\ {}^{g_k}E^{f_k}(r) \\ &\le \liminf_{k\to\infty}\ {}^{g_k}E^{h_k}(r) \\ &\le \limsup_{k\to\infty}(1 + c\lambda_k)\ E^{h_k}(r) \\ &= E^{h_*}(r) \\ &= E^{f_*}(r). \end{aligned}$$

Here, the last line follows from the fact that $f_* = h_*$ by Proposition 49. Since there is no loss of total energy, we see that the directional energies converge by using the lower semicontinuity. Q.E.D.

Lemma 61 *Let g be a normalized Lipschitz metric on $\mathbf{B}(r)$ and $f : (\mathbf{B}(r), g) \to (Y, d)$ a harmonic map. Then its tangent map $f_* : \mathbf{B}(1) \to Y_*$ is homogeneous of order α where α is the order of f at 0.*

PROOF. Follows immediately from Lemma 50 and Lemma 60. Q.E.D.

Lemma 62 *Let g be a normalized Lipschitz metric on $\mathbf{B}(r)$, $f : (\mathbf{B}(r), g) \to (Y, d)$ a harmonic map and $f_* : \mathbf{B}(1) \to Y_*$ its tangent map. For every $\beta, r' \in (0, 1)$, there exists B so that*

$$d_*(f_*(x), f_*(y)) \le B|x - y|^{\beta}$$

for all $x, y \in \mathbf{B}(r')$ and B is only depending on the choice of β, r', $\mathbf{B}(1)$ and the total energy of f_*.

PROOF. First, note that h_k converges to f_* uniformly by Proposition 49. Next, note that the energy of h_k converges to that of f_* by Lemma 60. Thus, the result follows from Lemma 58. Q.E.D.

Lemma 63 *Let g be a normalized Lipschitz Riemannian metric on $\mathbf{B}(r)$ and $f : (\mathbf{B}(r), g) \to (Y, d)$ a harmonic map. Then the order α of f at 0 is ≥ 1.*

PROOF. Since f_* is homogeneous of degree α,

$$d_*(f_*(tx), f_*(0)) = |tx|^\alpha d_*(f_*(\frac{x}{|x|}), f_*(0)).$$

On the other hand, for any $\beta \in (0,1)$ and t small, there exists a constant B so that

$$d_*(f_*(tx), f_*(0)) \leq B|tx|^\beta$$

by Lemma 62. Thus,

$$d_*(f_*(\frac{x}{|x|}), f_*(0)) \leq B|tx|^{\beta-\alpha}.$$

If $\alpha < 1$, choose β so that $\beta > \alpha$ and take the limit as $t \to 0$ to obtain

$$d_*(f_*(\frac{x}{|x|}), f_*(0)) = 0.$$

Since the choice of $x \in \mathbf{B}(1)$ is arbitrary, this contradicts the fact that f_* is non-constant (cf. Lemma 47). Q.E.D.

Using the fact that the order at a point on D is ≥ 1, we can prove Lipschitz continuity in $\mathbf{B}(1)$.

Theorem 64 *Let $\mathbf{B}$ be a dimension-n, codimension-1 local model, g a Lipschitz Riemannian metric on $\mathbf{B}(r)$, (Y,d) an NPC space and $f : (\mathbf{B}(r), g) \to (Y,d)$ a harmonic map. Then f is Lipschitz in $\mathbf{B}(\varrho r)$ with Lipschitz constant depending on $\varrho \in (0,1)$, $(\mathbf{B}(r), g)$ and the total energy E^f of f.*

PROOF. For each $x \in \mathbf{B}(1)$, the normalized map $f \circ L_x$ (cf. Proposition 6) has order ≥ 1 at 0 by Lemma 63. Thus, the order α_x of f at x is ≥ 1. The result now follows from Theorem 40. Q.E.D.

8.3 At a higher codimension singular point

Now we consider a dimension-n, codimension-ν local model where $\nu \geq 2$. Generally, we do not expect a harmonic map from this space to be Lipschitz continuous. On the other hand, we show that Lipschitz continuity can be proved with an additional assumption.

First, we establish some properties of the tangent map. Lemmas 65 and 66 and 67 below are the analogues of Lemmas 59, 60 and 55 corresponding to the codimension-ν case for $\nu \geq 2$.

Lemma 65 *Let $\mathbf{B}$ be a dimension-n, codimension-ν local model with $\nu \geq 2$, g a normalized Lipschitz metric defined on $\mathbf{B}(r)$ and $f : (\mathbf{B}(r), g) \to (Y,d)$ a* *harmonic map. Suppose $h_k : (\mathbf{B}(1), \delta) \to (Y, d)$, $f_k : (\mathbf{B}, g) \to (Y, d)$, $h_* = f_* : \mathbf{B}(1) \to (Y, d)$ defined as in Section 6. Let D_ϵ be the ϵ-neighborhood of D and $D_\epsilon(r) = \mathbf{B}(r) \cap D_\epsilon$. Fix $R \in (0,1)$. For any $r \in (0, R)$, there exist C and $\kappa > 0$, k_0 sufficiently large and $\epsilon_0 > 0$ sufficiently small so that*

$$^{\delta}E^{h_k}[D_\epsilon(r)] \le C\epsilon^\kappa, \ \forall k > k_0, \epsilon < \epsilon_0.$$

Proof. As in the proof of Lemma 59, there exists a constant c_3 so that

$$\frac{\epsilon E_x^{h_k}(\epsilon)}{I_x^{h_k}(\epsilon)} \le c_3.$$

Thus, by Theorem 41,

$$E_x^{h_k}(\epsilon) \le \frac{c_3 I_x^{h_k}(\epsilon)}{\epsilon} \le \frac{c_3 C^2 \epsilon^{2\gamma+n-1}}{\epsilon} =: c_4 \epsilon^{2\gamma+n-2}.$$

We can cover $D_\epsilon(r)$ be $\frac{c_5}{\epsilon^{n-\nu}}$ number of (2ϵ)-balls. Thus,

$$E^{h_k}[D_\epsilon(r)] \le \frac{c_5}{\epsilon^{n-\nu}} c_4 \epsilon^{2\gamma+n-2} =: c_6 \epsilon^{2\gamma-2+\nu}.$$

The lower semicontinuity of energy implies that

$$E^{f_*}[D_\epsilon(r)] \le c_6 \epsilon^{2\gamma-2+\nu}.$$

Q.E.D.

Lemma 66 *Let $h_k, f_k, h_* = f_*$ be as in Lemma 65. For $r \in (0,1)$,*

$$\lim_{k \to 0} {}^{\delta}E^{h_k}(r) = {}^{\delta}E^{h_*}(r)$$

and

$$\lim_{k \to \infty} {}^{\delta}E_{g_k}^{f_k}(r) = {}^{\delta}E^{f_*}(r).$$

Furthermore, the directional energies of f_k, h_k converge to that of f_*. The maps f_* is a homogeneous map of order α, where α is the order of f at 0.

Proof. Using Lemma 65, we can follow the argument of the proof of Lemma 60. Q.E.D.

Lemma 67 *Let $\mathbf{B}$ be a dimension-n, codimension-ν local model, g a normlized Lipschitz metric defined on $\mathbf{B}(r)$, (Y, d) an NPC space, $f : (\mathbf{B}(r), g) \to (Y, d)$ a harmonic map and $f_* : \mathbf{B}(1) \to Y_*$ its tangent map. Then for x, y on the same wedge at a distance ρ away from $D(1)$, we have*

$$d(f_*(x), f_*(y)) \le L|x - y|$$

where L is depending only on r, $\mathbf{B}(1)$ and ${}^{\delta}E^h$.

PROOF. If $x, y \in \mathbf{B}(1/2)$ are two points on the same wedge at a distance ρ away from $D(1)$, then Lemma 55 implies

$$d(h_k(x), h_k(y)) \leq L|x-y|$$

where c is depending on r and ${}^{\delta}E^{h_k}$. Thus, the result follows from the uniform convergence of h_k to f_* and the convergence of the energy of h_k to that of f_*. Q.E.D.

Lemma 68 *Let* $\mathbf{B}$ *be a dimension-*n*, codimension-*ν *local model,* g *a normalized Lipschitz metric defined on* $\mathbf{B}(r)$ *and* $f : (\mathbf{B}(r), g) \to (Y, d)$ *a harmonic map. Then its tangent map* $f_* : \mathbf{B}(1) \to Y_*$ *is homogeneous of order* α *where* α *is the order of* f *at 0.*

PROOF. Follows immediately from Lemma 50 and Lemma 66. Q.E.D.

Our next goal is to relate the order (and hence the Hölder exponent) of a harmonic map to the first eigenvalue associated with the domain and the target space. We start with a general definition of the first eigenvalue. Let G be a Riemannian complex with volume form ds and T an NPC space. The center of mass of a map $\varphi \in L^2(G, T)$ is a point $\bar{\varphi} \in T$ so that

$$\int_G d_T^2(\varphi, \bar{\varphi})ds = \inf_{P \in Y} \int_G d_T^2(\varphi, P)ds.$$

The unique existence of such a point is guaranteed by the NPC condition (cf. [KS1] Proposition 2.5.4). Now let $\mathcal{G}(T)$ be the set of Lipschitz maps $\varphi : G \to T$ into an NPC space T and define the first eigenvalue of G with values in T as

$$\lambda_1(G, T) = \inf_{\mathcal{G}(T)} \frac{\int_G |\nabla \varphi|^2 \, ds}{\int_G d_T^2(\varphi, \bar{\varphi})ds}. \tag{77}$$

In the applications, G will be a spherical complex associated with the domain of the map and the NPC space T will be a tangent cone of the target NPC space Y. The following results appear in [DM3] in the case when the domain is of dimension 2.

Lemma 69 *Suppose* $f : \mathbf{B}(r) \to (Y, d)$ *is a bounded map,* $\sigma \in (0, r)$ *and* $Q \in Y$ *so that*

$$\int_{\partial \mathbf{B}(\sigma)} d^2(f, Q)d\Sigma = \inf_{P \in Y} \int_{\partial \mathbf{B}(\sigma)} d^2(f, P)d\Sigma.$$

If $\pi : Y \to T_QY$ *is the projection map into the tangent cone of* Y *at* Q*, then*

$$\int_{\partial \mathbf{B}(\sigma)} d_{T_QY}^2(\pi \circ f, 0)d\Sigma = \inf_{V \in T_QY} \int_{\partial \mathbf{B}(\sigma)} d_{T_QY}^2(\pi \circ f, V)d\Sigma,$$

where 0 *is the origin of* T_QY*.*

PROOF. Let $t \mapsto c(t)$ be a geodesic so that $c(0) = Q$. By the minimizing property of $c(0) = Q$, we have

$$0 \leq \int_{\partial \mathbf{B}(\sigma)} d^2(f, c(t)) d\Sigma - \int_{\partial \mathbf{B}(\sigma)} d^2(f, c(0)) d\Sigma.$$

Furthermore, by [BH] Corollary II 3.6, we have

$$\lim_{t \to 0} \frac{d(f(y), c(t)) - d(f(y), c(0))}{t} = -\cos \angle(c, \gamma_y)$$

where γ_y is the geodesic from $c(0)$ to $f(y)$ and $\angle(\gamma_y, c)$ is the angle between γ_y and c at $c(0) = Q$. Therefore,

$$\begin{aligned}
0 &\leq \lim_{t \to 0} \int_{\partial \mathbf{B}(\sigma)} \frac{d^2(f, c(t)) - d^2(f, c(0))}{t} d\Sigma \\
&= \lim_{t \to 0} \int_{\partial \mathbf{B}(\sigma)} \frac{d(f, c(t)) - d(f, c(0))}{t} (d(f, c(t)) + d(f, c(0))) d\Sigma \\
&= -2 \int_{\partial \mathbf{B}(\sigma)} \cos \angle(\gamma_y, c) d(f, c(0)) d\Sigma.
\end{aligned}$$

Let $[c]$ be the equivalence class of c and $V = ([c], 1) \in T_Q Y$. Since $\pi \circ \gamma_y$ is the (radial) geodesic from the origin 0 to $\pi \circ f(y)$ in $T_Q Y$,

$$\cos \angle(\gamma_y, c) d(f(y), f(0)) = < \pi \circ f(y), V >,$$

and thus

$$0 \leq - \int_{y \in \partial \mathbf{B}(\sigma)} < \pi \circ f(y), V > d\Sigma. \tag{78}$$

By the continuity of the inner product, (78) holds for all $V = (V_0, t) \in T_Q Y$ where $V_0 = V/|V|$. Therefore, for $t \geq 0$,

$$\begin{aligned}
&\int_{\partial \mathbf{B}(\sigma)} d^2_{T_Q Y}(\pi \circ f(y), (V_0, t)) d\Sigma \\
&\quad = \int_{\partial \mathbf{B}(\sigma)} t^2 + |\pi \circ f(y)|^2 - 2t < \pi \circ f(y), V_0 > d\Sigma \\
&\quad \geq \int_{\partial \mathbf{B}(\sigma)} |\pi \circ f(y)|^2 d\Sigma \\
&\quad = \int_{\partial \mathbf{B}(\sigma)} d^2_{T_Q Y}(\pi \circ f(y), 0) d\Sigma.
\end{aligned}$$

Q.E.D.

Corollary 70 *Suppose* $f : \mathbf{B}(r) \to (Y, d)$ *is a bounded map,* $\sigma \in (0, r)$ *and* $Q \in Y$ *so that*

$$\int_{\partial \mathbf{B}(\sigma)} d^2(f, Q) d\Sigma = \inf_{P \in Y} \int_{\partial \mathbf{B}(\sigma)} d^2(f, P) d\Sigma.$$

If $\pi : Y \to T_Q Y$ *is the projection map into the tangent cone of* Y *at* Q *and* $\sigma : \mathbf{B}(1) \to \mathbf{B}(\sigma)$ *is defined by* $\sigma(x) = \sigma x$*, then*

$$\frac{\int_{\partial \mathbf{B}(1)} |\nabla^{\partial}(\pi \circ f \circ \sigma)(x)|^2 d\Sigma}{\int_{\partial \mathbf{B}(1)} |\pi \circ f \circ \sigma(x)|^2 d\Sigma} \geq \lambda_1(\partial \mathbf{B}(1), T_Q Y),$$

where ∇^{∂} *indicates that we are taking the tangential part of the energy density function on* $\partial \mathbf{B}(1)$*.*

PROOF. By Lemma 69, the center of mass of the map $\pi \circ f \circ \sigma$ is 0. Thus, the assertion follows immediately from the definition of $\lambda_1(\mathbf{B}(1), T_Q Y)$. Q.E.D.

A consequence of Corollary 70 is the following theorem which associates the first eigenvalue with the order of a harmonic map.

Theorem 71 *Let* $\mathbf{B}$ *be a dimension-*n*, codimension-*ν *local model,* g *a normalized Lipschitz Riemannian metric defined on* $\mathbf{B}(r)$*,* (Y, d) *an NPC space and* $f : (\mathbf{B}(r), g) \to (Y, d)$ *be a harmonic map. If* $\lambda_1(\partial \mathbf{B}(1), T_Q Y) \geq \beta$ *(*$> \beta$*) for all* $Q \in Y$ *then* $\alpha(\alpha + n - 2) \geq \beta$ *(*$> \beta$*), where* α *is the order of* f *at 0.*

PROOF. By inequalities (64), (65) and (68), it suffices to assume that the volume form and the directional derivatives are with respect to the Euclidean metric δ. This assumption is clearly without any loss of generality since in this proof we are interested in rescalings of $\mathbf{B}(\sigma)$ to unit size as $\sigma \to 0$ and at small scales the metric g is approximately Euclidean. We emphasize that f is harmonic with respect to the metric g which is not necessarily Euclidean.

Let $\sigma_i \to 0$ so that $f_{\sigma_i} \to f_* : \mathbf{B}(1) \to Y_*$. From Lemma 66, there exists λ so that

$$\lim_{\sigma_i \to 0} \int_{\partial \mathbf{B}(\lambda)} |\nabla^{\partial} f_{\sigma_i}|^2 d\Sigma = \int_{\partial \mathbf{B}(\lambda)} |\nabla^{\partial} f_*| d\Sigma.$$

By [GS], pages 200-201, we have

$$\lim_{\sigma \to 0} \frac{\sigma\lambda E(\sigma\lambda)}{\int_{\partial \mathbf{B}(\sigma\lambda)} d^2(f, Q_{\sigma\lambda}) d\Sigma} = \lim_{\sigma \to 0} \frac{\sigma\lambda E(\sigma\lambda)}{\int_{\partial \mathbf{B}(\sigma\lambda)} d^2(f, f(0)) d\Sigma},$$

where $Q_\sigma \in Y$ is the point so that

$$\int_{\partial \mathbf{B}(\sigma)} d^2(f, Q_\sigma) d\Sigma = \inf_{Q \in Y} \int_{\partial \mathbf{B}(\sigma)} d^2(f, Q) d\Sigma.$$

This then implies

$$\lim_{\sigma\to 0}\frac{\int_{\partial\mathbf{B}(\sigma\lambda)} d^2(f,Q_{\sigma\lambda})d\Sigma}{\int_{\partial\mathbf{B}(\sigma\lambda)} d^2(f,f(0))d\Sigma}=1. \tag{79}$$

Let $Q_i := Q_{\sigma_i} \in Y$ and $\pi_i : Y \to T_{Q_i}Y$ be a projection map into the tangent cone of Y at Q_i. By Lemma 69,

$$\int_{\partial\mathbf{B}(\sigma_i\lambda)} d^2(\pi_i\circ f,0)d\Sigma=\inf_{V\in T_{Q_i}Y}\int_{\partial\mathbf{B}(\lambda\sigma_i)} d^2(\pi_i\circ f,V)d\Sigma. \tag{80}$$

Additionally,

$$d^2(f,Q_i)=|\pi_i\circ f|^2 \;\text{ and } |\nabla^\partial f|^2\geq|\nabla^\partial(\pi_i\circ f)|^2 \tag{81}$$

since π_i is distance non-increasing. Thus, by (79) and (81),

$$\begin{aligned}
\lim_{\sigma_i\to 0}\frac{\lambda^2\int_{\partial\mathbf{B}(\lambda)}|\nabla^\partial f_{\sigma_i}|^2 d\Sigma}{\int_{\partial\mathbf{B}(\lambda)} d^2_{\sigma_i}(f_{\sigma_i},f_{\sigma_i}(0))d\Sigma} &= \lim_{\sigma_i\to 0}\frac{(\sigma_i\lambda)^2\int_{\partial\mathbf{B}(\sigma_i\lambda)}|\nabla^\partial f|^2 d\Sigma}{\int_{\partial\mathbf{B}(\sigma_i\lambda)} d^2(f,f(0))d\Sigma}\\
&= \lim_{\sigma_i\to 0}\frac{(\sigma_i\lambda)^2\int_{\partial\mathbf{B}(\sigma_i\lambda)}|\nabla^\partial f|^2 d\Sigma}{\int_{\partial\mathbf{B}(\sigma_i\lambda)} d^2(f,Q_i)d\Sigma}\\
&\geq \lim_{\sigma_i\to 0}\frac{(\sigma_i\lambda)^2\int_{\partial\mathbf{B}(\sigma_i\lambda)}|\nabla^\partial(\pi_i\circ f)|^2 d\Sigma}{\int_{\partial\mathbf{B}(\sigma_i\lambda)} |\pi_i\circ f|^2d\Sigma}.
\end{aligned}$$

By change of coordinates $y=\sigma_i\lambda x$, (80) and Corollary 70,

$$\begin{aligned}
\frac{(\sigma_i\lambda)^2\int_{y\in\partial\mathbf{B}(\sigma_i\lambda)}|\nabla^\partial(\pi_i\circ f)(y)|^2 d\Sigma}{\int_{\partial\mathbf{B}(\sigma_i\lambda)}|\pi_i\circ f(y)|^2 d\Sigma} &= \frac{\int_{x\in\partial\mathbf{B}(1)}|\nabla^\partial(\pi_i\circ f\circ(\sigma_i\lambda))(x)|^2 d\Sigma}{\int_{x\in\partial\mathbf{B}(1)}|(\pi_i\circ f\circ(\sigma_i\lambda))(x)|^2 d\Sigma}\\
&= \frac{\int_{x\in\mathbf{B}(1)}|\nabla^\partial(\pi_i\circ f\circ(\sigma_i\lambda)|^2(x) d\Sigma}{\int_{x\in\mathbf{B}(1)}|(\pi_i\circ f\circ(\sigma_i\lambda))(x)|^2 d\Sigma}\\
&\geq \lambda_1(\partial\mathbf{B}(1),T_{Q_i}Y)\\
&\geq \beta(>\beta).
\end{aligned}$$

Therefore,

$$\begin{aligned}
R &:= \frac{\int_{\partial\mathbf{B}(1)}|\nabla^\partial f_*|^2 d\Sigma}{\int_{\partial\mathbf{B}(1)} d^2(f_*,f_*(0))d\Sigma}\\
&= \frac{\lambda^2\int_{\partial\mathbf{B}(\lambda)}|\nabla^\partial f_*|^2 d\Sigma}{\int_{\partial\mathbf{B}(\lambda)} d^2(f_*,f_*(0))d\Sigma}\\
&= \lim_{\sigma_i\to 0}\frac{\lambda^2\int_{\partial\mathbf{B}(\lambda)}|\nabla^\partial f_{\sigma_i}|^2 d\Sigma}{\int_{\partial\mathbf{B}(\lambda)} d^2_{\sigma_i}(f_{\sigma_i},f_{\sigma_i}(0))d\Sigma}\\
&\geq \beta(>\beta).
\end{aligned}$$

For $y \in \partial\mathbf{B}(1)$, the homogeneity of f_* implies

$$d(f_*(ry), f_*(0)) = r^\alpha d(f_*(y), f_*(0)),$$

and hence

$$\begin{aligned} E^{f_*}(1) &= \int_{y\in\partial\mathbf{B}(1)} \int_0^1 \left(\left| \frac{\partial f_*}{\partial r}(ry) \right|^2 + \frac{1}{r^2} |\nabla^\partial f_*(ry)|^2 \right) r^{n-1} dr d\Sigma \\ &= \int_{y\in\partial\mathbf{B}(1)} \int_0^1 \left(\alpha^2 r^{2\alpha+n-3} d^2(f_*(y), f_*(0)) + r^{2\alpha+n-3} |\nabla^\partial f_*(y)|^2 \right) dr d\Sigma \\ &= \frac{\alpha^2}{2\alpha+n-2} \int_{y\in\partial\mathbf{B}(1)} d^2(f_*(y), f_*(0)) d\Sigma + \frac{1}{2\alpha+n-2} \int_{y\in\partial\mathbf{B}(1)} |\nabla^\partial f_*(y)|^2 d\Sigma. \end{aligned}$$

Thus,

$$\alpha = \frac{E^{f_*}(1)}{I^{f_*}(1)} = \frac{\alpha^2}{2\alpha+n-2} + \frac{1}{2\alpha+n-2} R$$

and

$$\alpha(\alpha+n-2) = R \geq \beta (> \beta).$$

Q.E.D.

Given $\mathbf{B}$ and any $x \in \mathbf{B}(r)$, consider $f \circ L_x : \mathbf{B}'_x(r(x)) \to (Y, d)$ of Proposition 6 where $\mathbf{B}'_x$ is a local model associated with the point x. We let

$$\lambda_1 := \inf_{x\in\mathbf{B}(r), Q\in Y} \lambda_1(\partial\mathbf{B}'_x(1), T_Q Y).$$

Corollary 72 *Let $\mathbf{B}$ be a dimension-n, codimension-ν local model, g a normalized Lipschitz Riemannian metric defined on $\mathbf{B}(r)$ and (Y, d) an NPC space. If $\lambda_1 \geq \alpha(\alpha+n-2)$ and $f : (\mathbf{B}(r), g) \to (Y, d)$ is a harmonic map, then f is Hölder continuous with Hölder exponent α in $\mathbf{B}(\varrho r)$ for $\varrho \in (0, 1)$.*

PROOF. For any $x \in \mathbf{B}(\varrho r)$, Theorem 71 says that the assumption $\lambda_1 \geq \alpha(\alpha+n-2)$ for all $Q \in Y$ implies the order of f at x is $\geq \alpha$. The result now follows from Theorem 40. Q.E.D.

We now give a sufficient condition implying that the order of a harmonic map is ≥ 1. For each $x \in D$, let $N(x)$ be the ν-plane perpendicular to D at x. Note that $|x| < 1$ implies that $\partial\mathbf{B}(1) \cap N(x)$ is a spherical $(\nu-1)$-complex. We first need the following lemma.

Lemma 73 *Let $f_* : \mathbf{B}(1) \to Y_*$ be a tangent map of a harmonic map. If the order of f_* is not equal to 1, then f_* is constant in the direction parallel to D.*

PROOF. By Lemma 51, we may assume that f_* maps into a cone with $f_*(0)$ equal to the vertex. Also, we may assume by homogeneity of f_* that the domain of f_* is $\mathbf{B}$. Let $x, y \in \mathbf{B}(1)$ be points on the same wedge and same distance to D and $x' = tx$, $y' = ty$. Then

$$t^\alpha d_*(f_*(x), f_*(y)) = d_*(f_*(x'), f_*(y')) \leq L|x' - y'| = Ct$$

where L is the Lipschitz constant of f_* and C is a constant depending on L and on the angle between the line from x to 0 and y to 0 respectively. Thus,

$$d_*(f_*(x), f_*(y)) \leq Ct^{1-\alpha}.$$

and the Lemma follows by letting $t \to 0$ if $\alpha < 1$ or $t \to \infty$ if $\alpha > 1$. Q.E.D.

Theorem 74 *Let $\mathbf{B}$ be a dimension-n, codimension-ν local model, g a normalized Lipschitz Riemannian metric defined on $\mathbf{B}(r)$, (Y, d) an NPC space and $f : (\mathbf{B}(r), g) \to (Y, d)$ be a harmonic map. If $\lambda_1(\partial\mathbf{B}(1) \cap N(0), T_Q Y) \geq \beta (> \beta)$ and $\alpha < 1$ for all $Q \in Y$, then the order α of f at 0 satisfies $\alpha(\alpha + \nu - 2) \geq \beta (> \beta)$.*

PROOF. By the homogeneity of f_*,

$$\begin{aligned} \alpha &= \frac{E^{f_*}(1)}{I^{f_*}(1)} \\ &= \frac{\int_{\mathbf{B}(1)} |\nabla f_*|^2 d\mu}{\int_{\partial\mathbf{B}(1)} d^2(f_*, 0) d\Sigma} \\ &= \frac{\int_{x\in D} \int_{y\in\mathbf{B}(1)\cap N(x)} |\nabla f_*|^2 dy\ dx}{\int_{x\in D} \frac{1}{(1-|x|^2)^{1/2}} \int_{\partial\mathbf{B}(1)\cap N(x)} d^2(f_*, 0) d\Sigma\ dx}. \end{aligned} \tag{82}$$

We use the notation ∇^N to indicate the we are taking the directional energy of f_* on $\partial\mathbf{B}(1) \cap N(x)$. Using Lemma 73, $|\nabla f_*|^2 = |\nabla^N f_*|^2$ and hence

$$\begin{aligned} &\int_{y\in\mathbf{B}(1)\cap N(x)} |\nabla f_*|^2(y) dy \\ &= \int_{y\in\mathbf{B}(1)\cap N(x)} |\nabla^N f_*|^2(y) dy \\ &= \int_{y\in\mathbf{B}\left((1-|x|^2)^{1/2}\right)\cap N(0)} |\nabla^N f_*|^2(y) dy \\ &= (1-|x|^2)^{\alpha-1} \int_{y\in\mathbf{B}\left((1-|x|^2)^{1/2}\right)\cap N(0)} |\nabla^N f_*|^2((1-|x|^2)^{-1/2} y) dy. \end{aligned}$$

Here, the second equality follows from translation in direction parallel to D and the last equality follows from the homogeneity of f_*.

Now apply the change of coordinates $z = (1-|x|^2)^{-1/2}y$ to obtain

$$\begin{aligned}
&\int_{y\in \mathbf{B}\left((1-|x|^2)^{1/2}\right)\cap N(0)} |\nabla^N f_*|^2((1-|x|^2)^{-1/2}y)dy \\
&= (1-|x|^2)^{\frac{\nu}{2}} \int_{\mathbf{B}(1)\cap N(0)} |\nabla^N f_*|^2(z)dz.
\end{aligned}$$

Hence the numerator in (82) is

$$\int_{x\in D} (1-|x|^2)^{\frac{\nu-2+2\alpha}{2}} dx \int_{\mathbf{B}(1)\cap N(0)} |\nabla^N f_*|^2(z)dz.$$

Similarly, the denominator of (82) is

$$\begin{aligned}
&\int_{x\in D} \frac{1}{(1-|x|^2)^{1/2}} dx \int_{\partial\mathbf{B}(1)\cap N(x)} d^2(f_*,0)d\Sigma \\
&= \int_{x\in D} (1-|x|^2)^{\frac{\nu-2+2\alpha}{2}} dx \int_{\partial\mathbf{B}(1)\cap N(0)} d^2(f_*,0)d\Sigma.
\end{aligned}$$

Thus, as in the proof of Theorem 71, we obtain

$$\begin{aligned}
\alpha &= \frac{\int_{\mathbf{B}(1)\cap N(0)} |\nabla^N f_*|^2 dy}{\int_{\partial\mathbf{B}(1)\cap N(0)} d^2(f_*,0)d\Sigma} \\
&= \frac{\alpha^2}{2\alpha+\nu-2} + \frac{R}{2\alpha+\nu-2}
\end{aligned}$$

and hence

$$\alpha(\alpha+\nu-2) = R \geq \beta (> \beta).$$

Q.E.D.

Given $\mathbf{B}$ and any $x \in \mathbf{B}(r)$, consider $f \circ L_x : \mathbf{B}'_x(r(x)) \to (Y,d)$ of Proposition 6 where $\mathbf{B}'_x$ is a local model associated with the point x. We let

$$\lambda_1^N := \inf_{x\in\mathbf{B}(r), Q\in Y} \lambda_1(\partial\mathbf{B}'_x(1)\cap N(0), T_Q Y).$$

Corollary 75 *Let $\mathbf{B}$ be a dimension-n, codimension-ν local model, g a Lipschitz Riemannian metric defined on $\mathbf{B}(r)$ and (Y,d) an NPC space. If $\lambda_1^N \geq \nu-1$ and $f : \mathbf{B}(r) \to (Y,d)$ is a harmonic map, then f is Lipschitz continuous in $\mathbf{B}(\varrho r)$ for $\varrho \in (0,1)$.*

PROOF. For any $x \in \mathbf{B}(\varrho r)$, Theorem 74 says that the assumption $\lambda_1^N \geq \nu-1$ implies that the order of f at x is ≥ 1. The result now follows from Theorem 40. Q.E.D.

9 Main Theorem

Here, we collect the regularity results from the previous sections to summarize our main regularity theorem for Lipschitz Riemannian complexes.

Theorem 76 *Let $B(r)$ be a ball of radius r around a point x in an admissible complex X endowed with a Lipschitz Riemannian metric g, (Y,d) an NPC space and $f:(B(r),g)\to(Y,d)$ a harmonic map.*

(1) If $x\in X-X^{(n-2)}$, let d denote the distance of x to $X^{(n-2)}$. Then for $\varrho\in(0,1)$ and $d'\le\min\{\varrho r,\varrho d\}$, f is Lipschitz continuous in $B(d')$ with Lipschitz constant depending on the total energy of the map f, $(B(r),g)$, d and ϱ.

(2) If $x\in X^{(k)}-X^{(k-1)}$ for $k=0,...,n-2$, let d denote the distance of x to $X^{(k-1)}$. Then for $\varrho\in(0,1)$ and $d'\le\min\{\varrho r,\varrho d\}$, f is Hölder continuous in $B(d')$ with Hölder exponent and constant depending on on the total energy of the map f, $(B(r),g)$, d and ϱ. More precisely, the Hölder exponent α has a lower bound given by the following: If $\lambda_1^N\ge\beta(>\beta)$ then $\alpha(\alpha+n-k-2)\ge\beta(>\beta)$. In particular, if $\lambda_1^N\ge n-k-1$, then f is Lipschitz continuous in a neighborhood of x.

PROOF. The assertion follows from Theorem 64, Theorem 74 and Corollary 75. Q.E.D.

References

[BH] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature.* Springer, 1991.

[BT] F. Bruhat and J. Tits. *Groupes réductifs sur un corps local. I Données radicielles valuées.* Publ. Math. IHES 41 (1972) 5-251.

[Ch] J. Chen. *On energy minimizing mappings between and into singular spaces.* Duke Math. J. 79 (1995), 77-99.

[Co] K. Corlette. *Archemedean superrigidity and hyperbolic geometry.* Ann. of Math. 135 (1992) 165-182.

[DM1] G. Daskalopoulos and C. Mese. *Harmonic maps from a 2-complex.* Communications in Analysis and Geometry 14 (2006) 1-52.

[DM2] G. Daskalopoulos and C. Mese. *Harmonic maps from a simplicial complex and geometric rigidity.* J. Diff. Geom. 78 (2008) 269-293.

[DM3] G. Daskalopoulos and C. Mese. *Fixed point and rigidity theorems for harmonic maps from simplicial complexes into NPC spaces.* Geom. Dedicata (2009) 141 33-57.

[EF] J. Eells and B. Fuglede. *Harmonic maps between Riemannian polyhedra.* Cambridge Tracts in Mathematics 142, Cambridge University Press, Cambridge 2001.

[F] B. Fuglede. *Hölder continuity of harmonic maps from Riemannian polyhedra to spaces of upper bounded curvature.* Calc. Var. Partial Differential Equations 16 (2003), no. 4, 375–403.

[GS] M. Gromov and R. Schoen. *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one.* Publ. Math. IHES 76 (1992) 165-246.

[IN] H. Izeki and S. Nayatani. *Combinatorial harmonic maps and discrete-group actions on Hadamard spaces.* Geom. Dedicata 114 (2005) 147-188.

[J] J. Jost. *Nonpositive curvature: geometric and analytic aspects.* Lectures in Mathematics. ETH Zürich, Birkhäuser Verlag 1997, MR1451625, Zbl 0896.53002.

[KS1] N. Korevaar and R. Schoen. *Sobolev spaces and harmonic maps for metric space targets.* Comm. Anal. Geom. 1 (1993) 561-659.

[KS2] N. Korevaar and R. Schoen. *Global existence theorem for harmonic maps to non-locally compact spaces.* Communications in Analysis and Geometry 5 (1997), 333-387.

[M1] C. Mese. *Regularity of harmonic maps from a flat complex.* Progress in Non-Linear Differential Equations. (2004) 133-148. Birkhauser-Verlag, Basel.

[M2] C. Mese *Harmonic maps into spaces with an upper curvature bound in the sense of Alexandrov.* Math. Z. 242 (2002), 633-661.

[Mo] C.B. Morrey. *Multiple integrals in the calculus of variations.* Springer-Verlag, New York - Berlin, 1966.

[Se] T. Serbinowski. *Boundary regularity of harmonic maps to nonpositively curved metric spaces.* Comm. Anal. Geom. 2 (1994) 139-154.

[Si] L. Simon. *Theorems on regularity and singularity of energy minimizing maps.* Birkhäuser, 1996

[W1] M.-T. Wang. *A fixed point theorem of discrete group actions on Riemannian manifolds.* J. Diff. Geom. 50 (1998) 249-267.

[W2] M.-T. Wang. *Generalized harmonic maps and representations of discrete groups.* Comm. Anal. Geom. 8 (2000) 545-563.