C^1 estimates for the Weil-Petersson Metric

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Abstract

We prove that the Weil-Petersson metric near the boundary of the Teichmüller space is C^1-asymptotically a product of the Weil-Petersson metric on a lower dimensional Teichmüller space and a metric on a model space. In particular, we show that the Weil-Petersson metric on the genus g, p-punctured Teichmüller space with $3g-3+p>0$ satisfies all the important properties required to apply the results in [DaMe1]. These estimates extend the well known C^0-estimates for the Weil-Petersson metric.

1 Introduction

The Teichmüller space $\mathcal{T}$ of a genus g, p-punctured surface S with $3g-3+p>0$ endowed with the Weil-Petersson metric G_{WP} is an incomplete Kähler manifold (cf. [Wo5] and [Chu]). Its metric completion $\overline{\mathcal{T}}$, although no longer a Riemannian manifold, is a CAT(0) space; i.e. a simply connected, complete metric space with non-positive curvature in the sense of Alexandrov (cf. [Ya]). Set theoretically, $\overline{\mathcal{T}}$ is the augmented Teichmuller space of Abikoff (cf. [Abi]). The boundary $\partial\mathcal{T}$ of Teichmüller space is stratified by lower dimensional Teichmüller spaces with each stratum being totally geodesic. In [Mas], Masur initiated the study of the Weil-Petersson metric near the boundary of $\overline{\mathcal{T}}$. In recent years, many authors have extended Masur's work to establish significant properties of the Weil-Petersson geometry ([Schu],

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[DaWe], [Ya], [Wo1], [Wo2], [Wo3], [Wo6], [Schu], [LSY1], [LSY2] among many others).

The C^0 estimates of [Ya], [DaWe], [Wo1] quantify the way in which $\mathcal{T}$ is asymptotically a product space of a lower dimensional Teichmüller space and its normal space near a point of the boundary. In this paper, we extend this result by proving the C^1 estimates. The motivation comes from the desire to use *differential* geometric methods in the study of Teichmüller space and its mapping class group. The estimates proven here are more delicate than the derivative estimates of [LSY1] and [LSY2] in the sense that we estimate the asymptotic *difference* of the Weil-Petersson metric and the product space. Significantly, the asymptotic C^1 estimates of this paper are needed to establish the *Holomorphic Rigidity Theorem* of Teichmüller space of [DaMa3]. The holomorphic rigidity is a surprisingly strong statement about the uniqueness of the complex structure of Teichmüller space; indeed, it asserts that a Kähler manifold which allows an action of the mapping class group such that the quotient is of finite volume must be biholomorphic to the Teichmüller space (under some mild assumptions). This application will be summarized later in this section.

We now turn to a brief description of our results. A point in $\mathcal{T}$ is realized as a nodal surface R with nodes $\{n_1, \ldots, n_N\}$. These nodes results from pinching N disjoint, nonhomotopic, noncontractible simple closed curves on the surface S. Let R_0 be the punctured surface $R \backslash \{n_1, \ldots, n_N\}$. The boundary stratum of $\partial\mathcal{T}$ containing R_0 is a (possibly a product) Teichmuller space $\mathcal{T}'$ of dimension $n = 3g - 3 + p - N$. Let $s = (s_1, \ldots, s_n) \in \mathbf{C}^n \mapsto R_s$ be a parameterization of the neighborhood of R_0 in $\mathcal{T}'$. We can regularize each node n_i by the the plumbing construction of Earle-Marden and Fay to obtain a family of smooth surfaces. Let $t = (t_1, \ldots, t_N) \in \mathbf{C}^N$ be the *plumbing coordinates*; thus $t_i \in \mathbf{C}$ parameterizes the regularization of the node n_i, and the family of surfaces forms a node as $|t_i| \to 0$ (with a nontrivial loop degenerating to the node n_i). Together s and t define the coordinates of the deformation space near the original surface R. (For more details, see the beginning of Section 2 for the case of one node and its generalization to many nodes at the beginning Section 5.) We first state in the next theorem the well known C^0 estimates of the Weil-Petersson metric and co-metric (cf. [Ya], [DaWe], [Wo1]). For clarity, we will use the upper case I, J, K and the lower case i, j, k, l to index the s-coordinates and the t-coordinates respectively.

Theorem 1 *The Weil-Petersson metric* $G_{WP} = (G_{**})$ *and*

$$h_{i\bar{i}} = \pi^3 |t_i|^{-2}(-\log|t_i|)^{-3}$$

satisfy the following estimates:

$$\begin{array}{rrcl}
(i) & G_{i\bar{i}} & = & h_{i\bar{i}}\left(1 + O\left(\sum_{l=1}^{N}(-\log|t_l|)^{-2}\right)\right) \\
(ii) & G_{j\bar{k}} & = & O\left((-\log|t_j|)^{-3}(-\log|t_k|)^{-3}|t_j|^{-1}|t_k|^{-1}\right) \quad (j \neq k) \\
(iii) & G_{I\bar{j}} & = & O\left(|t_j|^{-1}(-\log|t_j|)^{-3}\right) \\
(iv) & G_{I\bar{J}} & = & G_{I\bar{J}}(0) + O\left(\sum_{l=1}^{q}(-\log|t_l|)^{-2}\right).
\end{array}$$

The Weil-Petersson co-metric $G_{WP}^{-1} = (G^{**})$ *and*

$$h^{i\bar{i}} = \frac{|t_i|^2(-\log|t_i|)^3}{\pi^3}$$

satisfy the following estimates:

$$\begin{array}{rrcl}
(i') & G^{i\bar{i}} & = & h^{i\bar{i}}\left(1 + O\left(\sum_{l=1}^{N}(-\log|t_l|)^{2}\right)\right) \\
(ii') & G^{j\bar{k}} & = & O(|t_j||t_k|) \quad (j \neq k) \\
(iii') & G^{I\bar{j}} & = & O(|t_j|) \\
(iv') & G^{I\bar{J}} & = & G_{I\bar{J}}(0) + O\left(\sum_{l=1}^{N}(-\log|t_l|)^{2}\right).
\end{array}$$

The main result of this paper asserts that the C^1 estimates of the Weil-Petersson co-metric is the "derivative" of error term of the C^0 estimates. More precisely, we have the following.

Theorem 2 *The Weil-Petersson co-metric* $G_{WP}^{-1} = (G^{**})$ *satisfies the following estimates (with* i, j, k, I, J, K *all distinct):*

$$(i) \quad \frac{\partial}{\partial t_i} G^{i\bar{i}} = \frac{\partial}{\partial t_i} h_{ii} + O\left(|t_i|(-\log|t_i|)\right)$$

$$
\begin{aligned}
(ii) \quad \frac{\partial}{\partial t_i} G^{j\bar{j}} &= O\left(|t_i|^{-1}(-\log|t_i|)^{-3}|t_j|^2(\log|t_j|)^3\right) \\
(iii) \quad \frac{\partial}{\partial t_i} G^{i\bar{j}} &= O\left(|t_j|\right) \\
(iv) \quad \frac{\partial}{\partial t_i} G^{j\bar{k}} &= O\left(|t_i|^{-1}(-\log|t_i|)^{-3}|t_j||t_k|\right) \\
(v) \quad \frac{\partial}{\partial t_i} G^{I\bar{J}} &= O\left(|t_i|^{-1}(-\log|t_i|)^{-3}\right) \\
(vi) \quad \frac{\partial}{\partial t_i} G^{I\bar{j}} &= O\left(|t_i|^{-1}(-\log|t_i|)^{-3}|t_j|\right) \\
(vii) \quad \frac{\partial}{\partial t_i} G^{I\bar{i}} &= O(1).
\end{aligned}
$$

By inverting the matrix $G^{i\bar{j}}$ and combining the above two theorems, we obtain the following.

Theorem 3 *The Weil-Petersson metric satisfies the following estimates:*

$$
\begin{aligned}
(i) \quad \frac{\partial}{\partial t_i} G_{i\bar{i}} &= \frac{\partial}{\partial t_i} h_{i\bar{i}} + O\left(|t_i|^{-3}(-\log|t_i|)^{-5}\right) \\
(ii) \quad \frac{\partial}{\partial t_i} G_{j\bar{j}} &= O\left(|t_i|^{-1}(-\log|t_i|)^{-3}|t_j|^{-2}(\log|t_j|)^{-3}\right) \\
(iii) \quad \frac{\partial}{\partial t_i} G_{i\bar{j}} &= O\left(|t_i|^{-2}(-\log|t_i|)^{-3}(|t_j|^{-1}(-\log|t_j|)^{-3}\right) \\
(iv) \quad \frac{\partial}{\partial t_i} G_{j\bar{k}} &= O\left(|t_i|^{-1}(-\log|t_i|^{-3})(|t_j|^{-1}(-\log|t_j|)^{-3}(|t_k|^{-1}(-\log|t_k|)^{-3}\right) \\
(v) \quad \frac{\partial}{\partial t_i} G_{I\bar{J}} &= O\left(|t_i|^{-1}(-\log|t_i|)^{-3}\right) \\
(vi) \quad \frac{\partial}{\partial t_i} G_{I\bar{j}} &= O\left(|t_i|^{-1}(-\log|t_i|)^{-3}|t_j|^{-1}(-\log|t_j|)^{-3}\right) \\
(vii) \quad \frac{\partial}{\partial t_i} G_{I\bar{i}} &= O\left(|t_i|^{-2}(-\log|t_i|)^{-3}\right).
\end{aligned}
$$

We would like to point out that there is a different approach in expressing C^1 estimates for the Weil-Petersson metric due to Scott Wolpert (cf. [Wo3]). In this work, Wolpert writes the Weil-Petersson connection in terms of a certain frame given by gradients of geodesic length functions, but unfortunately this frame does not come from a set of local coordinates on Teichmüller space. Even though such an approach is effective in terms of obtaining curvature estimates near the boundary of Teichmüller space, it is not clear to the authors how to use it in conjunction with harmonic maps. In other words, in order to obtain good estimates for harmonic maps we need to be able to write down local coordinate expressions. This is one of the reasons for carrying out this work.

We will now give an explicit description of the way the Weil-Petersson metric is a product metric and discuss the aforementioned application to the *Holomorphic Rigidity Theorem* of Teichmüller space proven in [DaMa3]. The key ingredient in the proof is the theory of harmonic maps which has played a central role in various geometric rigidity problems (e.g. [Siu], [MSY], [JoYa2], [Cor], [GrSc], etc.). Given that the Weil-Petersson curvature is strongly negative in the sense of Siu (cf. [Schu]), a natural way to prove rigidity properties in Teichmüller space is by applying Siu's Bochner method (cf. [Siu]). Jost and Yau conjectured that an equivariant harmonic map into $\overline{\mathcal{T}}$ must lie in the interior $\mathcal{T}$ (unless it maps completely to the boundary) and stated the holomorphic rigidity property of Teichmüller space (cf. [JoYa1]). In another direction, Farb-Masur and Yeung (cf. [FaMa] and [Ye]) established super-rigidity properties of the mapping class group providing further evidence of the Jost-Yau conjecture. We have so far been unable apriori (from only local properties) to verify that harmonic maps indeed map into $\mathcal{T} \subset \overline{\mathcal{T}}$. On the other hand, using the C^1-estimates of this paper, we show in a series of papers [DaMe1], [DaMe2] and [DaMa3] that the singular set of a harmonic map (i.e. the set of points that are not mapped into a single stratum of $\overline{\mathcal{T}}$) is small enough so that we can apply the Bochner method.

The starting point of this work is [DaMe1] where we study harmonic maps into a space that is asymptotically a product space. Theorem 1, Theorem 2 and Theorem 3 imply that the Weil-Petersson metric g_{WP} of $\mathcal{T}$ near the boundary $\partial\mathcal{T}$ is asymptotically a product of the Weil-Petersson metric g_{wp} on a lower dimensional Teichmüller space $\mathcal{T}'$ and its normal space. To make this statement more explicit, we recall the model space $(\mathbf{H}, h_{\mathbf{H}})$ where

$$\mathbf{H} = \{(r,\theta) \in \mathbf{R}^2 : r > 0\}, \quad h_{\mathbf{H}}(r,\theta) = 4dr^2 + r^6 d\theta^2$$

first introduced by Yamada (cf. [Ya]) (Note that in [DaMe2] and [DaMa3], we consider the slightly different metric $g_{\mathbf{H}} = d\rho^2 + \rho^6 d\phi^2$ which is clearly isometric to $h_{\mathbf{H}}$ via the change of coordinates $\rho = 2r, \phi = \frac{\theta}{8}$.) The Riemann surface $(\mathbf{H}, h_{\mathbf{H}})$ models the singular behavior of the Weil-Peterson metric. For example, the Gauss curvature of $(\mathbf{H}, h_{\mathbf{H}})$ approaches $-\infty$ as $r \to 0$. This corresponds to the sectional curvature blow up of the Weil-Petersson metric near $\partial\mathcal{T}$. Moreover, $(\mathbf{H}, h_{\mathbf{H}})$ is not complete; the curve $r \mapsto (r, \theta_0)$ for a fixed ϕ_0 leaves every compact subset of $\mathbf{H}$ as $r \to 0$. Recall that in [Wo5] and [Chu], it was shown that certain curves in $\mathcal{T}$ leave every compact subset have finite length. These correspond to deformations of compact Riemann surfaces via neck pinching. The metric completion of $(\mathbf{H}, h_{\mathbf{H}})$ is constructed by identifying the axis $r = 0$ to a single point P_0 and setting

$$\overline{\mathbf{H}} = \mathbf{H} \cup \{P_0\}.$$

The distance function $d_{\mathbf{H}}$ induced by $h_{\mathbf{H}}$ is extended to $\overline{\mathbf{H}}$ by setting $d_{\mathbf{H}}(Q, P_0) = r$ for $Q = (r, \theta) \in \mathbf{H}$. Consider the metric

$$h = h_{\mathbf{H}} \oplus \ldots \oplus h_{\mathbf{H}} \text{ defined on } \mathbf{H} \times \ldots \times \mathbf{H}$$

and the product space

$$(\mathcal{T}' \times \mathbf{H} \times \ldots \times \mathbf{H}, g_{wp} \oplus h)$$

where each copy of $\mathbf{H}$ corresponds to a neck pinching. Denote by (r_i, θ_i) the coordinates (r, θ) of $\mathbf{H}$ on the i^{th} copy in $\mathbf{H} \times \ldots \times \mathbf{H}$. The relation between the complex coordinate t_i and (r_i, θ_i) is given by

$$r_i = 2\pi^2(-\log|t_i|)^{-\frac{1}{2}}, \qquad \theta_i = \arg t_i.$$

The asymptotic C^0 product structure of the Weil-Petersson geometry can be described by

$$G_{WP} - G_{wp} \oplus h = O(|r|^3)h \tag{1}$$

(cf. [Ya], [DaWe] and [Wo1]). In particular, there exists a constant c such that given a point p close to the boundary in $\mathcal{T}' \times \mathbf{H} \times \ldots \times \mathbf{H}$ with coordinates (r_i, θ_i) on the i^{th} copy of $\mathbf{H}$, the quantity

$$|r| := \sqrt{\sum_{i=1}^{N} r_i^2}$$

is bounded by c times the Weil-Petersson distance of p to the boundary of Teichmüller space. Combining this with Theorem 1, Theorem 2, Theorem 3, the s-derivative estimates already derived in [Schu], [LSY1], [LSY1] and a change of coordinates, we obtain the following.

Corollary 4 *Given a point P in the boundary of Teichmüller space, there is a neighborhood $\mathcal{N} \simeq \mathcal{U} \times \mathcal{V} \subset \mathcal{T}' \times \overline{\mathbf{H}} \times \ldots \times \overline{\mathbf{H}}$ of P that metrically satisfies Assumption 2 of [DaMe1].*

To complete the proof of the *Holomorphic Rigidity Theorem*, note that the existence of equivariant harmonic maps from Riemannian domains to the Weil-Petersson completion $\overline{\mathcal{T}}$ of Teichmüller space $\mathcal{T}$ was established in [DaWe] provided that the action is sufficiently large. The idea is to show that u maps into a single stratum of $\overline{\mathcal{T}}$ outside a small set. Indeed, Corollary 4 allows us to use the techniques of [DaMe1] to prove that the singular set is of Hausdorff codimension 2. In [DaMa3], we show that this regularity result is sufficient to apply the Bochner method implying the holomorphic rigidity of Teichmüller space.

We end this section with a brief summary of the ideas in this paper. Central to this paper is Wolpert's grafted metric (cf. [Wo1]) defined on each of the surfaces obtained by the plumbing construction. In Section 2, we recall the grafted metric and obtain estimates for the grafting functions. In Section 3, we derive estimates for the t_i-derivatives for the grafted metrics and its curvature. In Section 4, we compare the grafted metric and the hyperbolic metric. The key is the elliptic equation (41) derived from the curvature identity (also used in [Wo1]). In order to take advantage of this elliptic equation, we introduce a global vector field defined on the deformation space of the original Riemann surface that projects down to the vector field $\frac{\partial}{\partial t_i}$ (where t_i comes from the plumbing coordinates). With this global formulation, we can use the maximum principle to derive estimates for the comparison function of the grafted and hyperbolic metrics. Combined with results from Section 3, we thereby obtain t_i-derivative estimates for the hyperbolic metric. Section 5 contains the proof of the main results. The Weil-Petersson co-metric can be written down as an integral involving the Masur differentials (cf. [Mas]) and the hyperbolic metric. Using estimates from the previous sections, we finally derive the co-metric estimates of Theorem 2.

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2 Wolpert's grafted metric

In this section, we derive some estimates associated with Wolpert's grafted metric (cf. [Wo1]). Let R be a nodal surface possibly with punctures and a single node n and let $R_0 = R\backslash\{n\}$ be the Riemann surface with additional punctures $\{a, b\}$. Let g_0^{hyp} be the complete hyperbolic metric on R_0 and

$$u_0, v_0 : \mathcal{U}, \mathcal{V} \to \mathbf{D}^* = \{0 < |z| < 1\}$$

be cusp coordinates on open sets $\mathcal{U}$, $\mathcal{V}$ near a, b respectively (we assume that $\mathcal{U}$, $\mathcal{V}$ contain exactly one puncture). In other words, g_0^{hyp} in $\mathcal{U}$, $\mathcal{V}$ is given in the coordinates $\zeta = u, v$ respectively as

$$g_0^{hyp}(\zeta) = h_0(\zeta)|d\zeta|^2 = \left(\frac{|d\zeta|}{|\zeta|\log|\zeta|}\right)^2.$$

We can parameterize a neighborhood of R_0 in the deformation space $\mathrm{Def}(R_0)$ by Beltrami differentials. More specifically, fix a basis consisting of Beltrami differentials $\nu_1, \dots, \nu_n$ (where $n = \dim \mathrm{Def}(R_0)$) and let R_s be the surface whose complex structure is defined by

$$\nu = \nu(s) = \sum_{k=1}^{n} s_k \nu_k, \;\; s = (s_1, \dots, s_n) \in \mathbf{C}^n \tag{2}$$

with $|s|$ small via the Beltrami equation. Then $s \to R_s$ defines a parametrization of a neighborhood of R_0 in $\mathrm{Def}(R_0)$. For later purposes, we choose ν_i ($i = 1, \dots, n$) with support disjoint from $\mathcal{U} \cup \mathcal{V}$.

Let g_s^{hyp} be the complete hyperbolic metric on R_s and u_s, v_s be cusp coordinates in $\mathcal{U}$, $\mathcal{V}$ near a, b respectively. Define

$$f_s := u_0 \circ u_s^{-1} \;\text{ and }\; g_s := v_0 \circ v_s^{-1}.$$

By the removable singularity theorem and by multiplying f_s and g_s by $(f_s'(0))^{-1}$ and $(g_s'(0))^{-1}$ respectively, we can assume that

$$f_s(0) = 0,\ f_s'(0) = 1,\ g_s(0) = 0,\ g_s'(0) = 1. \tag{3}$$

Furthermore, since we have chosen $\nu(s)$ to have support in a set disjoint from $\mathcal{U} \cup \mathcal{V}$, we have that

$$f_s, g_s \text{ are bi-holomorphic onto their image (cf. [Wo1] 2.4.M).} \tag{4}$$

For $|t|$ small, we denote by $R_{s,t}$ the Riemann surface obtained by the plumbing construction. In other words, we remove punctured discs from R_s and glue back an annulus via the plumbing equation $u_0 v_0 = t$. We can rewrite this equation as

$$(f_s \circ u_s) \cdot (g_s \circ v_s) = t. \tag{5}$$

Note that since $\nu(s)$ is supported away from $\mathcal{U} \cup \mathcal{V}$, the discs that we remove can be chosen to be the same for all s. The parameter t is called the *plumbing coordinate*.

The following subsets of the Riemann surface $R_{s,t}$ are defined as in [Wo1]. Let $A \in (0, \frac{1}{2})$ such that $A < |f_s| < 2A$ and $A < |g_s| < 2A$ are relatively compact annuli in $\mathcal{U}$ and $\mathcal{V}$ respectively. For $\delta > 0$ small, let

$$\begin{array}{rcl} I_\delta^{s,t} & := & \{Ae^{-2\delta} < |f_s| < 2Ae^{2\delta}\} = \{\frac{|t|}{2Ae^{2\delta}} < |g_s| < \frac{|t|}{Ae^{-2\delta}}\} \\ II_\delta^{s,t} & := & \{\frac{|t|}{Ae^{2\delta}} < |f_s| < Ae^{2\delta}\} = \{\frac{|t|}{Ae^{2\delta}} < |g_s| < Ae^{2\delta}\} \\ III_\delta^{s,t} & := & \{\frac{|t|}{2Ae^{2\delta}} < |f_s| < \frac{|t|}{Ae^{-2\delta}}\} = \{Ae^{-2\delta} < |g_s| < 2Ae^{2\delta}\} \\ II_{1,\delta}^{s,t} & := & \{|t|^{\frac{1}{2}+2\delta} < |f_s| < Ae^{2\delta}\} = \{\frac{|t|}{Ae^{2\delta}} < |g_s| < |t|^{\frac{1}{2}-2\delta}\} \\ II_{2,\delta}^{s,t} & := & \{\frac{|t|}{Ae^{2\delta}} < |f_s| < |t|^{\frac{1}{2}-2\delta}\} = \{|t|^{\frac{1}{2}+2\delta} < |g_s| < Ae^{2\delta}\}. \end{array}$$

Definition 5 By considering a lift of the deformation space $\mathrm{Def}(R_0)$ of the original Riemann surface R_0 to Teichmüller space, one obtains a family of Riemann surfaces

$$F : \mathcal{R} \to \mathcal{S}$$

whose fiber over (s,t) is $R_{s,t}$.

The local coordinates u_s, v_s of $R_{s,t}$ glue together to define local coordinates u, v along the fibers of $\mathcal{R}$. We set

$$\mathcal{N}^1 = \cup_{(s,t)\in\mathcal{S}} II_{1,\delta}^{s,t},\ \ \mathcal{N}^2 = \cup_{(s,t)\in\mathcal{S}} II_{2,\delta}^{s,t} \ \text{ and } \ \mathcal{N} = \mathcal{N}^1 \cup \mathcal{N}^2. \tag{6}$$

Thus, we obtain local coordinates (u, s, t) and (v, s, t) for $\mathcal{N}$. We will use the coordinates (u, s, t) in $\mathcal{N}^1$ and the coordinates (v, s, t) in $\mathcal{N}^2$. The coordinates (u, s, t) are used in $\mathcal{N}^1 \cap \mathcal{N}^2$.

Definition 6 Let $g_{s,t}^{hyp}$ be the hyperbolic metric on $R_{s,t}$. We write with respect to local coordinates

$$g_{s,t}^{hyp} = \rho|d\zeta|^2.$$

We now follow Wolpert [Wo1] to construct a metric on $R_{s,t}$ conformal to $g_{s,t}^{hyp}$.

STEP 1: *Region near* $|\zeta| = A$. In $(I_\delta^{s,t} \cap II_{1,\delta}^{s,t}) \cup (III_\delta^{s,t} \cap II_{2,\delta}^{s,t})$, we graft $g_s^{hyp} = h_0(\zeta)|d\zeta|^2 = \left(\frac{|d\zeta|}{|\zeta|\log|\zeta|}\right)^2$ with

$$\begin{aligned} h_t(\zeta)|d\zeta|^2 &= \left(\frac{\pi}{\log|t|}\csc(\frac{\pi\log|\zeta|}{\log|t|})\frac{|d\zeta|}{|\zeta|}\right)^2 \qquad (7) \\ &= \theta^2\csc^2\theta\ h_0(\zeta)|d\zeta|^2 \quad \text{where} \quad \theta = \frac{\pi\log|\zeta|}{\log|t|}. \end{aligned}$$

More precisely, the grafted metric is given by

$$h_0^{\eta(\zeta)}(\zeta)\ h_t^{1-\eta(\zeta)}(\zeta)|d\zeta|^2$$

in $\mathcal{N}^1$ with $\zeta = u$ (resp. in $\mathcal{N}^2$ with $\zeta = v$) where η is a smooth function of $\alpha = \log|\zeta|$ with $\eta \equiv 0$ for $|\zeta| \leq Ae^{-\delta}$ and $\eta \equiv 1$ for $|\zeta| \geq Ae^{\delta}$. This is [Wo1] 3.4.MG (model grafting).

STEP 2: *Region near* $|\zeta| = |t|^{\frac{1}{2}}$. Since the conformal structure on $R_{s,t}$ is determined by the identification $f_s(u_s)g_s(v_s) = t$, the metrics $h_t(u_s)|du_s|^2$ and $h_t(v_s)|dv_s|^2$ do not agree on $II_{1,\delta}^{s,t} \cap II_{2,\delta}^{s,t}$ (unless of course $f_s(u) = u$ and $g_s(v) = v$). Thus, we construct a new metric by grafting $h_t(u_s)|du_s|^2$ and $h_t(v_s)|dv_s|^2$ by a smooth function $\eta = \eta(\alpha)$, $\alpha = \log|u_s|$ with $\eta \equiv 0$ for $|u_s| \geq |t|^{\frac{1}{2}-\delta}$ and $\eta \equiv 1$ for $|u_s| \leq |t|^{\frac{1}{2}+\delta}$. This is [Wo1] 3.4.CG (compound grafting).

Definition 7 The *grafted metric* constructed above will be denoted $g_{s,t}^{gr}$. We write in local coordinates

$$g_{s,t}^{gr} =: \omega|d\zeta|^2. \qquad (8)$$

To understand this grafting, Wolpert also considers the auxiliary metrics

$$\begin{aligned} h_{t,aux}(u_s,t)|du_s|^2 &:= \left(\frac{\pi}{\log|t|}\csc(\frac{\pi\log|f_s(u_s)|}{\log|t|})\frac{|f_s'(u_s)||du_s|}{|f_s(u_s)|}\right)^2 \\ h_{t,aux}(v_s,t)|dv_s|^2 &:= \left(\frac{\pi}{\log|t|}\csc(\frac{\pi\log|g_s(v_s)|}{\log|t|})\frac{|g_s'(v_s)||dv_s|}{|g_s(v_s)|}\right)^2 \end{aligned} \tag{9}$$

which are compatible with the identification $f_s(u_s)g_s(v_s) = t$ and hence they define a metric on $II_\delta^{s,t}$.

Remark 8 The grafting region of Step 1 is the set

$$\{\zeta \in \mathcal{N}^i : Ae^{-\delta} \le |\zeta| \le Ae^{\delta}\}, \quad i = 1, 2.$$

The grafting region of Step 2 is (assuming that $|s|$ is sufficiently small)

$$\{\zeta \in \mathcal{N}^1 : |t|^{\frac{1}{2}-\delta} \le |u| \le |t|^{\frac{1}{2}+\delta}\} \subset \mathcal{N}^1 \cap \mathcal{N}^2.$$

Since $f_0(u) = u$ and $g_0(v) = v$, the normalization (3) implies that

$$f_s(u) = u + O(u^2) \;\text{ and }\; g_s(v) = v + O(v^2) \tag{10}$$

where the error terms $O(u^2)$ and $O(v^2)$ can be chosen independently of s for $|s|$ small. Using $f_s(u)g_s(v) = t$ (cf. the plumbing equation (5)), we can view u as a function of v and t (resp. v as a function of u and t). Combined with (10), we obtain

$$t = uv + O(u^2v) + O(uv^2) \tag{11}$$

and hence

$$u = O(\frac{|t|}{|v|}), \quad v = O(\frac{|t|}{|u|}), \quad \frac{\partial u}{\partial v} = O(\frac{|t|}{|v|^2}), \quad \frac{\partial v}{\partial u} = O(\frac{|t|}{|u|^2}), \tag{12}$$

$$\frac{\partial^2 u}{\partial v^2} = O(\frac{|t|}{|v|^3}), \quad \frac{\partial^2 v}{\partial u^2} = O(\frac{|t|}{|u|^3}), \quad \frac{\partial u}{\partial t} = O(\frac{1}{|v|}), \quad \frac{\partial v}{\partial t} = O(\frac{1}{|u|}). \tag{13}$$

Definition 9 In $\mathcal{N}^1$ (resp. $\mathcal{N}^2$) with respect to the coordinates (u, s, t) (resp. (v, s, t)), define the functions

$$\begin{aligned} \rho(\zeta, s, t) := \rho_{s,t}(\zeta), &\qquad \omega(\zeta, s, t) := \omega_{s,t}(\zeta), \\ h(\zeta, t) := h_t(\zeta), &\qquad h_{aux}(\zeta, t) := h_{t,aux}(\zeta) \end{aligned}$$

for $\zeta = u$ (resp. $\zeta = v$).

In particular,

$$h(\zeta,t) = \left(\frac{\theta \csc\theta}{|\zeta|\log|\zeta|}\right)^2, \quad \theta = \frac{\pi \log|\zeta|}{\log|t|}$$

Lemma 10 *In $\mathcal{N}^1$ (resp. $\mathcal{N}^2$), we have with respect to the coordinates (u,s,t) (resp. (v,s,t)),*

$$\frac{\frac{\partial h}{\partial t}}{h} = \frac{1}{t(-\log|t|)}(1-\theta\cot\theta) \quad \text{and} \quad \frac{\frac{\partial h}{\partial \zeta}}{h} = -\frac{1}{\zeta}\left(\frac{\theta\cot\theta}{\log|\zeta|+1}\right)$$

where $\theta = \frac{\pi\log|\zeta|}{\log|t|}$ and $\zeta = u$ (resp. $\zeta = v$). In particular,

$$\frac{\frac{\partial h}{\partial t}}{h} = O(|t|^{-1}(-\log|t|)^{-3})(-\log|u|)^2 \quad \text{in } \mathcal{N}^1 \text{ (resp. } \mathcal{N}^2\text{)}$$

and

$$\frac{\frac{\partial h}{\partial \zeta}}{h} = O(|u|^{-1}(-\log|u|)^{-1}) \quad \text{in } \mathcal{N}^1\cap\mathcal{N}^2.$$

Proof. The identities follow by a straightforward calculation. Note that $1-\theta\cot\theta = O(\theta^2)$ in $\mathcal{N}^1$ (resp. $\mathcal{N}^2$) and $\cot\theta$ is bounded in $\mathcal{N}^1\cap\mathcal{N}^2$. Q.E.D.

In order to study the grafted metric $\omega|d\zeta|^2$, we introduce the function

$$\psi(\zeta,t) := \frac{1}{2}\log\left(\frac{h(\zeta,t)}{h_0(\zeta)}\right) = \log(\theta\csc\theta), \quad \theta = \frac{\pi\log|\zeta|}{\log|t|} \tag{14}$$

for $\zeta = u$ (resp. $\zeta = v$) in $\mathcal{N}^1$ (resp. $\mathcal{N}^2$). Recall that f (resp. g) is close to the identity. Thus, in $\mathcal{N}^1$ where $|t|^{\frac{1}{2}-2\delta} < |f_s(u)| < Ae^{2\delta}$ (resp. in $\mathcal{N}^2$ where $|t|^{\frac{1}{2}-2\delta} < |g(v)| < Ae^{2\delta}$) for $\delta > 0$ small and $A < 1$, the function θ ranges from a value slightly less than $\frac{\pi}{2}$ to a value stricly less than π.

Lemma 11 *In $\mathcal{N}^1$ (resp. $\mathcal{N}^2$), we have with respect to the coordinates (u,s,t) (resp. (v,s,t)),*

$$\begin{aligned}
2\frac{\partial\psi}{\partial t} &= 2\frac{\frac{\partial h}{\partial t}}{h} = \frac{1}{t(-\log|t|)}(1-\theta\cot\theta)\\
2\frac{\partial\psi}{\partial\zeta} &= \frac{1}{\zeta\log|\zeta|}(1-\theta\cot\theta)\\
4\frac{\partial^2\psi}{\partial\zeta\partial t} &= \frac{1}{t\zeta(\log|t|)(\log|\zeta|)}(\theta\cot\theta-\theta^2\csc^2\theta)
\end{aligned}$$

where $\zeta = u$ (resp. $\zeta = v$).

PROOF. The identities follow from (14) and a straightforward calculation. Q.E.D.

Lemma 12 *In $\mathcal{N}^1$ (resp. $\mathcal{N}^2$) whenever $Ae^{-\delta} < |u|$ with respect to the coordinates (u, s, t) (resp. $Ae^{-\delta} < |v|$ with respect to the coordinates (v, s, t)), we have*

$$\begin{aligned} \psi &= O((-\log|t|)^{-2}) \\ \frac{\partial \psi}{\partial t} &= \frac{\frac{\partial h}{\partial t}}{h} = O(|t|^{-1}(-\log|t|)^{-3}) \\ \frac{\partial \psi}{\partial \zeta} &= O((-\log|t|)^{-2}) \\ \frac{\partial^2 \psi}{\partial t \partial \zeta} &= O(|t|^{-1}(-\log|t|)^{-3})) \end{aligned}$$

where $\zeta = u$ (resp. $\zeta = v$).

PROOF. We can apply Taylor expansions to conclude that $1 - \theta \cot \theta = O(\theta^2)$ and $\theta^2 \csc^2 \theta - \theta \cot \theta = O(\theta^2)$. For $Ae^{-\delta} < |u|$ (resp. $Ae^{-\delta} < |v|$), we have that $\log|u|$ (resp. $\log|v|$) are bounded functions. Thus assertion follows from (14) and Lemma 11. Q.E.D.

In order to study the grafted metric $\omega|du|^2$ near $|u| = |t|^{\frac{1}{2}}$ and $\omega|dv|^2$ near $|v| = |t|^{\frac{1}{2}}$ (i.e. in the grafting region of STEP 2), we introduce the functions

$$\Psi_1(u,t) := \frac{1}{2} \log\left(\frac{h(u,t)}{h_{aux}(u,t)}\right), \quad \Psi_2(v,t) := \frac{1}{2} \log\left(\frac{h(v,t)}{h_{aux}(v,t)}\right)$$

in $\mathcal{N}^1$ and $\mathcal{N}^2$ respectively. Writing $v = v(u)$ via the identification $f(u)g(v) = t$, define

$$\Psi_2(u,t) = \Psi_2(v(u),t) \tag{15}$$

as a function on $\mathcal{N}^1$. From [Wo1] p.442,

$$\Psi_i = O(|t|^{\frac{1}{2}-2\delta}) \text{ in } \mathcal{N}^1 \cap \mathcal{N}^2. \tag{16}$$

We will also need the derivative estimates of Ψ.

Lemma 13 *In $\mathcal{N}^1 \cap \mathcal{N}^2$, with respect to the coordinates (u, s, t),*

$$\frac{\partial \Psi_i}{\partial u} = O(|t|^{-4\delta}), \qquad \frac{\partial \Psi_i}{\partial t} = O(|t|^{-\frac{1}{2}-4\delta}),$$
$$\frac{\partial^2 \Psi_i}{\partial u^2} = O(|t|^{-\frac{1}{2}-8\delta}), \qquad \frac{\partial^2 \Psi_i}{\partial u \partial t} = O(|t|^{-1-10\delta})$$

for $i = 1, 2$.

PROOF. In $\mathcal{N}^1 \cap \mathcal{N}^2$,

$$h_{aux}(u,t) = \left(\frac{\theta \csc \Theta}{u \log |u|} \left| \frac{u f'(u)}{f(u)} \right| \right)^2, \quad \Theta = \frac{\pi \log |f(u)|}{\log |t|}$$

and

$$h(u,t) = \left(\frac{\theta \csc \theta}{|u| \log |u|} \right)^2, \quad \theta = \frac{\pi \log |u|}{\log |t|}.$$

Thus, we can write

$$\Psi_1(u,t) = \log(\sin \Theta) - \log(\sin \theta) - \log \left| \tilde{f}(u) \right|$$

where

$$\tilde{f}(u) = \frac{u f'(u)}{f(u)}.$$

Thus, by a straightforward computation,

$$\frac{\partial \Theta}{\partial u} = \frac{\pi}{2 \log |t| u} \tilde{f}(u), \quad \frac{\partial \theta}{\partial u} = \frac{\pi}{2 \log |t| u},$$

$$\begin{aligned}
\frac{\partial_1}{\partial u} &= \frac{\partial}{\partial u} \log(\sin \Theta) - \frac{\partial}{\partial u} \log(\sin \theta) - \frac{\partial}{\partial u} \log \left| \tilde{f}(u) \right| \\
&= \cot \Theta \cdot \frac{\partial \Theta}{\partial u} - \cot \theta \frac{\partial \theta}{\partial u} - \frac{\tilde{f}'(u)}{2 \tilde{f}(u)} \\
&= \frac{\pi}{2 \log |t| u} (\cot \Theta - \cot \theta) + \frac{\pi}{2 \log |t| u} \cot \Theta \left(\tilde{f}(u) - 1 \right) - \frac{\tilde{f}'(u)}{2 \tilde{f}(u)}.
\end{aligned}$$

and

$$\frac{\partial^2\Psi_1}{\partial u^2} = -\frac{1}{u}\left(\frac{\pi}{2\log|t|u}(\cot\Theta-\cot\theta)+\frac{\pi}{2\log|t|u}\cot\Theta\left(\tilde{f}(u)-1\right)\right)$$
$$+\left(\frac{\pi}{2\log|t|u}\right)^2(\cot'\Theta-\cot'\theta)+\left(\frac{\pi}{2\log|t|u}\right)^2\cot'\Theta\left((\tilde{f}(u))^2-1\right)$$
$$+\frac{\pi}{2\log|t|u}\cot\Theta\tilde{f}'(u)-\left(\frac{\tilde{f}'(u)}{2\tilde{f}(u)}\right)'.$$

By (10), $f(u)=u+O(|u|^2)$ is an analytic function which in turn implies $\tilde{f}(u)=\frac{uf'(u)}{f(u)}=1+O(|u|)$ is an analytic function. We thus obtain

$$\log\left|\frac{f(u)}{u}\right|=O(|u|),\ \tilde{f}(u)-1=O(|u|),\ \ \frac{\tilde{f}'(u)}{\tilde{f}(u)}=O(1) \tag{17}$$

and

$$|\Theta-\theta|=\left|\frac{\pi}{\log|t|}\log\left|\frac{f(u)}{u}\right|\right|=O(|u|). \tag{18}$$

In $\mathcal{N}^1\cap\mathcal{N}^2$ where $|t|^{\frac{1}{2}+2\delta}<|f_s(u)|<|t|^{\frac{1}{2}-2\delta}$, both Θ and θ are close to $\frac{\pi}{2}$. Thus,

$$\begin{array}{ll} |\cot\Theta|\le c & |\cot\Theta-\cot\theta|\le c|\Theta-\theta| \\ |\cot'\Theta|\le c & |\cot'\Theta-\cot'\theta|\le c|\Theta-\theta| \end{array}$$

where c denotes a constant independent of u,t and s. Combined with (17) and (18), we conclude $\frac{\partial\Psi_1}{\partial u}=O(1)$ and $\frac{\partial^2\Psi_1}{\partial u^2}=O(|t|^{-\frac{1}{2}-2\delta})$. The estimate $\frac{\partial\Psi_2}{\partial v}=O(1)$ and $\frac{\partial^2\Psi_2}{\partial v^2}=O(|t|^{-\frac{1}{2}-2\delta})$ follows from an analogous argument. By (12) and (13), $\frac{\partial v}{\partial u}=O(|t|^{-\frac{1}{2}-4\delta})$ and $\frac{\partial^2 v}{\partial u^2}=O(|t|^{-\frac{1}{2}-6\delta})$ in $\mathcal{N}^1\cap\mathcal{N}^2$ where $|t|^{\frac{1}{2}+2\delta}<|f(u)|<|t|^{\frac{1}{2}-2\delta}$. Thus,

$$\frac{\partial\Psi_2}{\partial u}=\frac{\partial\Psi_2}{\partial v}\frac{\partial v}{\partial u}=O(|t|^{-4\delta})$$

and

$$\frac{\partial^2\Psi_2}{\partial u^2}=\frac{\partial^2\Psi_2}{\partial v^2}\left(\frac{\partial v}{\partial u}\right)^2+\frac{\partial\Psi}{\partial v}\frac{\partial^2 v}{\partial u^2}=O(|t|^{-\frac{1}{2}-8\delta}).$$

By a straightforward calculation and with $q(x) = x\cot x$,

$$\frac{\partial \Psi_1}{\partial t} = \frac{\partial \Theta}{\partial t}\cot\Theta - \frac{\partial \theta}{\partial t}\cot\theta = \frac{1}{2t(-\log|t|)}\left(q(\Theta) - q(\theta)\right)$$

and

$$\begin{aligned}\frac{\partial^2 \Psi_1}{\partial u \partial t} &= \frac{1}{2t(-\log|t|)}\left(q'(\Theta)\frac{\partial \Theta}{\partial u} - q'(\theta)\frac{\partial \theta}{\partial u}\right) \\ &= \frac{\pi}{-2ut(\log|t|)^2}\left(\tilde{f}(u)q'(\Theta) - q'(\theta)\right) \\ &= \frac{\pi}{-2ut(\log|t|)^2}\left(q'(\Theta) - q'(\theta) + \left(\tilde{f}(u) - 1\right)q'(\Theta)\right).\end{aligned}$$

Noting that $q(x)$ is a smooth function near $x = \frac{\pi}{2}$, we conclude as before that $\frac{\partial \Psi_1}{\partial t} = O(|t|^{-\frac{1}{2}-4\delta})$ and $\frac{\partial^2 \Psi_1}{\partial u \partial t} = O(|t|^{-1})$. The estimate $\frac{\partial \Psi_2}{\partial t} = O(|t|^{-\frac{1}{2}-2\delta})$ follows by an analogous argument. Furthermore, since $\frac{\partial v}{\partial t} = O(|t|^{-\frac{1}{2}-2\delta})$, $\frac{\partial v}{\partial u} = O(|t|^{-4\delta})$, $\frac{\partial v}{\partial t} = O(|t|^{-\frac{1}{2}-2\delta})$, $\frac{\partial^2 v}{\partial u \partial t} = O(|t|^{-1-4\delta})$ and $\frac{\partial^2 v}{\partial v \partial t} = O(|t|^{-1-4\delta})$ in $\mathcal{N}^1 \cap \mathcal{N}^2$ by (13), we obtain

$$\frac{\partial \Psi_2}{\partial t} = \frac{\partial \Psi_2}{\partial v}\frac{\partial v}{\partial t} = O(|t|^{-\frac{1}{2}-4\delta}).$$

and

$$\frac{\partial^2 \Psi_2}{\partial u \partial t} = \frac{\partial^2 \Psi_2}{\partial v^2}\frac{\partial v}{\partial u}\frac{\partial v}{\partial t} + \frac{\partial \Psi_2}{\partial v}\frac{\partial^2 v}{\partial u \partial t} = O(|t|^{-1-10\delta}).$$

Q.E.D.

Definition 14 In $\mathcal{N}^1 \cap \mathcal{N}^2$, define with respect to the coordinates (u, s, t),

$$K_0 := h^{-\frac{1}{2}}\frac{\partial}{\partial u}, \quad L_0 := h^{-\frac{1}{2}}\frac{\partial}{\partial \bar{u}}, \quad D := \frac{1}{4}h^{-1}\frac{\partial^2}{\partial u \partial \bar{u}}. \tag{19}$$

Lemma 15 *In $\mathcal{N}^1 \cap \mathcal{N}^2$ with respect to the coordinates (u, s, t),*

$$K_0\Psi_i = O(|t|^{\frac{1}{2}-8\delta}),\ \frac{\partial}{\partial t}(K_0\Psi_i) = O(|t|^{-\frac{1}{2}-14\delta}),\ \frac{\partial}{\partial u}(K_0\Psi_i) = O(|t|^{-\frac{1}{2}-12\delta}),$$

$$\frac{\partial}{\partial \bar{t}}(K_0\Psi_i) = O(|t|^{-\frac{1}{2}-8\delta}),\ \frac{\partial}{\partial \bar{u}}(K_0\Psi_i) = O(|t|^{-\frac{1}{2}-12\delta})$$

for $i = 1, 2$.

PROOF. In $\mathcal{N}^1 \cap \mathcal{N}^2$ where $|t|^{\frac{1}{2}+2\delta} \le |f_s(u)| \le |t|^{\frac{1}{2}-2\delta}$, we have $h^{-\frac{1}{2}} = O(|t|^{\frac{1}{2}-4\delta})$. By Lemma 10,

$$\frac{\frac{\partial h}{\partial t}}{h} = O(|t|^{-1}) \text{ and } \frac{\frac{\partial h}{\partial u}}{h} = O(|t|^{-\frac{1}{2}-2\delta}).$$

Thus, Lemma 13 implies

$$K_0\Psi_i = h^{-\frac{1}{2}}\frac{\partial \Psi_i}{\partial u} = O(|t|^{\frac{1}{2}-8\delta}), \qquad \frac{\partial K_0}{\partial t}\Psi_i = -\frac{1}{2}\frac{\frac{\partial h}{\partial t}}{h}K_0\Psi_i = O(|t|^{-\frac{1}{2}-8\delta})$$

$$\frac{\partial K_0}{\partial u}\Psi_i = -\frac{1}{2}\frac{\frac{\partial h}{\partial u}}{h}K_0\Psi_i = O(|t|^{-10\delta}), \qquad K_0\frac{\partial \Psi_i}{\partial t} = h^{-\frac{1}{2}}\frac{\partial^2 \Psi_i}{\partial u \partial t} = O(|t|^{-\frac{1}{2}-14\delta})$$

$$K_0\frac{\partial \Psi_i}{\partial u} = h^{-\frac{1}{2}}\frac{\partial^2 \Psi_i}{\partial u^2} = O(|t|^{-12\delta}).$$

The assertion follows immediately from the above estimates. The $\bar{u}$ and $\bar{t}$ derivative estimates are proven similarly. Q.E.D.

Next, we derive estimates on the grafting function η of STEP 2.

Lemma 16 *Let η be as in [Wo1] 3.4.CG. In $\mathcal{N}^1 \cap \mathcal{N}^2$, with respect to the coordinates (u, s, t),*

$$\frac{\partial \eta}{\partial \bar{u}},\ \frac{\partial \eta}{\partial u} = |u|^{-1}O((-\log|t|)^{-1}),\ \frac{\partial^2 \eta}{\partial u^2},\ \frac{\partial^2 \eta}{\partial u \partial \bar{u}} = |u|^{-2}O((-\log|t|)^{-2})$$

$$\frac{\partial^3 \eta}{\partial u^2 \partial \bar{u}} = |u|^{-3}O((-\log|t|)^{-2}).$$

PROOF. Recall from [Wo1] 3.4.CG. that $\eta = \eta(a)$ where $a = \frac{\log|u|}{\log|t|}$. The C^k norm of η is t and s independent. Direct computation gives

$$\frac{\partial \eta}{\partial \bar{u}} = \eta'(a)\frac{\partial a}{\partial \bar{u}} = \eta'(a)\frac{1}{2\bar{u}\log|t|}, \qquad \frac{\partial \eta}{\partial u} = \eta'(a)\frac{\partial a}{\partial u} = \eta'(a)\frac{1}{2u\log|t|}.$$

$$\frac{\partial^2 \eta}{\partial u \partial \bar{u}} = \eta''(a)\frac{1}{4|u|^2(\log|t|)^2},\ \frac{\partial^2 \eta}{\partial u^2} = \eta''(a)\frac{1}{4u^2(\log|t|)^2} - \eta'(a)\frac{1}{2u^2\log|t|}$$

$$\frac{\partial^3 \eta}{\partial^2 u \partial \bar{u}} = \eta'''(a)\frac{1}{8u|u|^2(\log|t|)^3} - \eta''(a)\frac{1}{4u|u|^2(\log|t|)^2}.$$

The estimates follow immediately from the above identities. Q.E.D.

Lemma 17 *Let η be as in [Wo1] 3.4.CG. In $\mathcal{N}^1 \cap \mathcal{N}^2$ with respect to the coordinates* (u, s, t),

$$\begin{aligned} L_0\eta &= O(1), & D\eta &= O(1), \\ \frac{\partial L_0}{\partial t}\eta &= O(|t|^{-1}(\log|t|)^{-1}), & \frac{\partial D}{\partial t}\eta &= O(|t|^{-1}(\log|t|)^{-1}) \\ \frac{\partial}{\partial u}(L_0\eta) &= O(1) & \frac{\partial}{\partial u}(D\eta) &= O(1). \end{aligned}$$

Proof. Since $h^{-\frac{1}{2}} = O(\log|t|)$ in $\mathcal{N}_1 \cap \mathcal{N}_2$ by definition, we obtain from Lemma 16

$$\begin{aligned} L_0\eta &= h^{-\frac{1}{2}}\frac{\partial \eta}{\partial \bar{u}} = O(1), & D\eta &= h^{-1}\frac{\partial^2\eta}{\partial u \partial \bar{u}} = O(1), \\ L_0\frac{\partial \eta}{\partial u} &= h^{-\frac{1}{2}}\frac{\partial^2\eta}{\partial u\partial\bar{u}} = O((-\log|t|)^{-1}) & D\frac{\partial\eta}{\partial u} &= h^{-1}\frac{\partial^3\eta}{\partial u^2\partial\bar{u}} = O(1). \end{aligned}$$

Furthermore, by Lemma 10, $h^{-1}\frac{\partial h}{\partial t} = O(|t|^{-1}(-\log|t|)^{-1})$ and $h^{-1}\frac{\partial h}{\partial u} = O(1)$. Thus, we also obtain

$$\frac{\partial L_0}{\partial t}\eta = -\frac{1}{2}h^{-1}\frac{\partial h}{\partial t}L_0\eta = O(|t|^{-1}(\log|t|)^{-1}),$$

$$\frac{\partial D}{\partial t}\eta = -h^{-1}\frac{\partial h}{\partial t}D\eta = O(|t|^{-1}(\log|t|)^{-1}),$$

$$\frac{\partial L_0}{\partial u}\eta = -\frac{1}{2}h^{-1}\frac{\partial h}{\partial u}L_0\eta = O(1), \quad \frac{\partial D}{\partial u}\eta = -h^{-1}\frac{\partial h}{\partial u}D\eta = O(1).$$

The assertions follow immediately by combining the above estimates with Lemma 16. Q.E.D.

3 The derivative estimates

In this section, we derive derivative estimates for the grafted metric $g^{gr} = \omega|dz|^2$ and its curvature.

Lemma 18 *In $\mathcal{N}^1$ (resp. $\mathcal{N}^2$), we have in the coordinates (u,s,t) (resp. (v,s,t)),*

$$\frac{h}{\omega} = 1 + O((-\log|t|)^{-2}),$$
$$\frac{\partial}{\partial t}\left(\frac{h}{\omega}\right) = O(|t|^{-1}(-\log|t|)^{-3}).$$

and

$$\frac{\frac{\partial\omega}{\partial t}}{\omega} = O(|t|^{-1}(-\log|t|)^{-3})(-\log|\zeta|)^2.$$

Proof. We prove these estimates in $\mathcal{N}^1$. The same argument gives the estimates in $\mathcal{N}^2$. The proof consists of two steps.

(1) *We prove the estimates in $\mathcal{N}^1\backslash\mathcal{N}^2$ with respect to the coordinates (u,s,t).*

By Step 1 in the construction of the grafted metric that in $\mathcal{N}^1\backslash\mathcal{N}^2$,

$$\frac{h}{\omega} = \left(\frac{h}{h_0}\right)^{\eta} = e^{2\eta\psi} \quad\text{and}\quad \frac{\frac{\partial h}{\partial t}}{h} - \frac{\frac{\partial h}{\partial t}}{\omega} = \frac{\partial}{\partial t}\log\left(\frac{h}{\omega}\right) = 2\eta\frac{\partial\psi}{\partial t}$$

where ψ as in (14). Now the first and second estimates follow immediately from Lemma 12 and the fact that $\eta(\alpha(\zeta)) = 0$ whenever $|\zeta| < Ae^{-\delta}$. The third follows immediately from Lemma 10 and the second estimate.

(2) *We prove the estimates in $\mathcal{N}^1\cap\mathcal{N}^2$ with respect to the coordinates (u,s,t).*

By Step 2 in the construction of the grafted metric that in $\mathcal{N}^1\cap\mathcal{N}^2$,

$$\frac{h}{\omega} = e^{2\eta(\Psi_1-\Psi_2)} \quad\text{and}\quad \frac{\frac{\partial h}{\partial t}}{h} - \frac{\frac{\partial\omega}{\partial t}}{\omega} = \frac{\partial}{\partial t}\log\frac{h}{\omega} = 2\eta\left(\frac{\partial\Psi_1}{\partial t} - \frac{\partial\Psi_2}{\partial t}\right)$$

where η is a function of $\alpha = \log|u|$ (cf. [Wo1] 3.4.CG). Thus, the first and second estimates follow immediately by (16) and Lemma 13. The third estimate follows immediately from Lemma 10 and the second estimate. q.e.d.

Lemma 19 *In $\mathcal{N}^1\cap\mathcal{N}^2$ with respect to the coordinates (u,s,t),*

$$\frac{\frac{\partial h}{\partial u}}{h} - \frac{\frac{\partial\omega}{\partial u}}{\omega} = O(|t|^{-4\delta}).$$

PROOF. We have

$$\frac{\frac{\partial h}{\partial u}}{h} - \frac{\frac{\partial \omega}{\partial u}}{\omega} = \left(\frac{\partial}{\partial u}\log\frac{h}{\omega}\right) = 2\eta\frac{\partial(\Psi_1 - \Psi_2)}{\partial u} + 2(\Psi_1 - \Psi_2)\frac{\partial \eta}{\partial u}.$$

where η is a function of $\alpha = \log|u|$ (cf. [Wo1] 3.4.CG). The estimate follows immediately (16), Lemma 13 and Lemma 16. Q.E.D.

The Gauss curvature of the grafted metric $g^{gr}_{s,t}$ will be denoted $K^{gr}_{s,t}$. We now record Wolpert's estimates of curvature. By Remarks after Definition 3.8 and Lemma 3.9 of [Wo1], we have

$$K^{gr} + 1 = O((-\log|t|)^{-2}) \quad \text{in } \mathcal{N} \tag{20}$$

and

$$K^{gr} + 1 = O(|t|^{\frac{1}{2}-2\delta}) \quad \text{in } \mathcal{N}^1 \cap \mathcal{N}^2. \tag{21}$$

Furthermore, we will show

Lemma 20 *In $\mathcal{N}^1$ (resp. $\mathcal{N}^2$), we have in the coordinates (u,s,t) (resp. (v,s,t)),*

$$\frac{\partial K^{gr}}{\partial t} = O(|t|^{-1}(-\log|t|)^{-3}).$$

PROOF. We prove these estimates in $\mathcal{N}^1$. Similar argument gives the estimates in $\mathcal{N}^2$. The proof consists of two steps.

(1) *We prove the estimates in $\mathcal{N}^1 \backslash \mathcal{N}^2$ with respect to the coordinates (u,s,t).*

By [Wo1] p.441 we have

$$K^{gr} = -e^{-2\eta\psi}\left(1 + \psi\alpha^2\eta_{\alpha\alpha} + 2\alpha^2\eta_\alpha\psi_\alpha + \eta(e^{2\psi} - 1)\right) \tag{22}$$

where $\eta = \eta(\alpha)$ as in [Wo1] 3.4.MG, ψ as in (14) and the subscript in α denotes $\frac{d}{d\alpha}$. Differentiating with respect to t, we obtain

$$\frac{\partial K^{gr}}{\partial t} = -2\eta\frac{\partial \psi}{\partial t}K^{gr} - e^{-2\eta\psi}\left(\frac{\partial \psi}{\partial t}\alpha^2\eta_{\alpha\alpha} + 2\alpha^2\eta_\alpha\frac{\partial \psi_\alpha}{\partial t} + 2\eta\frac{\partial \psi}{\partial t}e^{2\psi}\right). \tag{23}$$

Recalling that $\log u = \alpha + i\beta$ (cf. [Wo1] 3.4.MG) and differentiating with respect to u, we obtain $\frac{\partial u}{\partial \alpha} = u$. Thus, Lemma 12 implies $\frac{\partial \psi_\alpha}{\partial t} = \frac{\partial^2 \psi}{\partial u \partial t} \frac{\partial u}{\partial \alpha} = O(|t|^{-1}(-\log|t|)^{-3})$. Since the function η is supported in $\{|u| \geq Ae^{-\delta}\}$ and its C^k estimate is independent of t and s, Lemma 12 and (20) imply that all terms in (23) are $O(|t|^{-1}(-\log|t|)^{-3})$.

(2) *We prove the estimate in* $\mathcal{N}^1 \cap \mathcal{N}^2$ *with respect to the coordinates* (u, s, t).

By [Wo1] p.438 formula (3.1),

$$K^{gr} = -e^{-2\eta\Psi}\left(1 + 4\Psi D\eta + 8\mathrm{Re}L_0\eta K_0\Psi + \eta(e^{2\Psi} - 1)\right). \qquad (24)$$

Let $t = t^1 + it^2$. Differentiating with respect to t^1, we obtain

$$\begin{aligned} \frac{\partial K^{gr}}{\partial t^1} &= -2\eta\frac{\partial \Psi}{\partial t^1}K^{gr} \\ &\quad -e^{-2\eta\Psi}\left(4\frac{\partial \Psi}{\partial t^1}D\eta + 4\Psi\frac{\partial D}{\partial t^1}\eta + 8Re\frac{\partial L_0}{\partial t^1}\eta K_0\Psi \right. \\ &\quad \left. +8\mathrm{Re}L_0\eta\frac{\partial}{\partial t^1}(K_0\Psi) + 2\frac{\partial \Psi}{\partial t^1}\eta e^{2\Psi}\right). \end{aligned} \qquad (25)$$

By estimates (16), (21), Lemma 13, Lemma 15 and Lemma 17 imply that all terms in (29) are $O(|t|^{-\beta})$ for some $\beta \in (0, 1)$. Since the same estimate holds for $\frac{\partial K^{gr}}{\partial t^2}$, we obtain the desired estimate. Q.E.D.

Lemma 21 *In* $\mathcal{N}^1 \backslash \mathcal{N}^2$ *(resp.* $\mathcal{N}^2 \backslash \mathcal{N}^1$*), we have in the coordinates* (u, s, t) *(resp.* (v, s, t)*),*

$$\frac{dK^{gr}}{d\zeta} = O((-\log|t|)^{-2}))$$

where $\zeta = u$ *(resp.* $\zeta = v$*). In* $\mathcal{N}^1 \cap \mathcal{N}^2$*, we have*

$$\frac{dK^{gr}}{d\zeta} = O(|t|^{-20\delta}).$$

Proof. We prove these estimates in $\mathcal{N}^1$. Similar argument gives the estimates in $\mathcal{N}^2$. The proof consists of two steps.

(1) *We prove the first estimate in* $\mathcal{N}^1 \backslash \mathcal{N}^2$ *with respect to the coordinates* (u,s,t).

Let $\log u = \xi = \alpha + i\beta$ and η as in [Wo2] 3.4 MG (cf. proof of Lemma 20). By differentiating (22) with respect to u

$$\begin{aligned}\frac{dK^{gr}}{du} &= -2\psi\frac{d\eta}{du}K^{gr} - 2\eta\frac{d\psi}{du}K^{gr} \\ &- e^{-2\eta\psi}\left(\frac{d}{du}(\psi\alpha^2\eta_{\alpha\alpha} + 2\alpha^2\eta_\alpha\psi_\alpha) + \frac{d\eta}{du}(e^{2\psi}-1) + 2\eta\frac{d\psi}{du}e^{2\psi}\right).\end{aligned} \tag{26}$$

In the region $\mathcal{N}^1 \backslash \mathcal{N}^2$, we have by Lemma 12

$$\psi = O((-\log|t|)^{-2}) \tag{27}$$

$$\frac{d\psi}{du} = O((-\log|t|)^{-2}) \tag{28}$$

and α, η and their derivatives are uniformly bounded. Hence the last term in (26) is $O((-\log|t|)^{-2})$. By (20), (27) and (28), the remaining two terms combine as

$$-2\eta\frac{d\psi}{du}K^{gr} - 2\eta\frac{d\psi}{du}e^{2\psi(1-\eta)} = -2\eta\frac{d\psi}{du}(K^{gr} + 1 - e^{2\psi(1-\eta)} - 1) = O((-\log|t|)^{-2}).$$

This proves the first estimate of the lemma.

(2) *We prove the estimate in* $\mathcal{N}^1 \cap \mathcal{N}^2$ *with respect to the coordinates* (u,s,t).

Let $u = u^1 + iu^2$. Differentiating (24) with respect to u^1, we obtain

$$\begin{aligned}\frac{\partial K^{gr}}{\partial u^1} &= -2\frac{\partial\eta}{\partial u^1}\Psi K^{gr} - 2\eta\frac{\partial\Psi}{\partial u^1}K^{gr} \\ &\quad -e^{-2\eta\Psi}\Bigg(4\frac{\partial\Psi}{\partial u^1}D\eta + 4\Psi\frac{\partial}{\partial u^1}(D\eta) + 8Re\frac{\partial}{\partial u^1}(L_0\eta)K_0\Psi \\ &\quad +8\mathrm{Re}L_0\eta\frac{\partial}{\partial u^1}(K_0\Psi) + \frac{\partial\eta}{\partial u^1}(e^{2\Psi}-1) + 2\frac{\partial\Psi}{\partial u^1}\eta e^{2\Psi}\Bigg).\end{aligned} \tag{29}$$

The estimates (16), (21), Lemma 13, Lemma 15, Lemma 16 and Lemma 17 and the fact that $|t|^{\frac{1}{2}-2\delta} \leq |f_s(u)| \leq |t|^{\frac{1}{2}+2\delta}$ imply that all terms in (29) are $O(|t|^{-20\delta})$. Since the same estimate holds for $\frac{\partial K^{gr}}{\partial u^2}$, we obtain the desired estimate.

Combining (1) and (2) proves the second estimate of the lemma. Q.E.D.

4 Comparison of hyperbolic and grafted metrics

In this section, we compare the hyperbolic metric $g^{hyp}_{s,t}$ and the grafted metric $g^{gr}_{s,t}$ defined on $R_{s,t}$. We will need to analyze the comparison function

$$\phi_{s,t} := \frac{1}{2} \log \frac{g^{hyp}_{s,t}}{g^{gr}_{s,t}}. \tag{30}$$

Expansion 4.2 of [Wo1] is

$$e^{2\phi_{s,t}} = 1 - \frac{\pi^2}{3}(-\log|t|)^{-2}(\triangle^{hyp}_{s,t} - 2)^{-1}\Lambda + O((-\log|t|)^{-4})$$

where $\triangle^{hyp}_{s,t}$ is the Laplacian with respect to the metric $g^{hyp}_{s,t}$ on $R_{s,t}$ and Λ is independent of t and its derivatives are bounded independently of s and supported in $(I^{s,t}_\delta \cap II^{s,t}_\delta) \cup (II^{s,t}_\delta \cap III^{s,t}_\delta)$. In particular, we have

$$e^{2\phi_{s,t}} = 1 + O((-\log|t|)^{-2}). \tag{31}$$

Let $\triangle^{gr}_{s,t}$ be the Laplacian on $R_{s,t}$ with respect to the grafted metric $g^{gr}_{s,t}$. Define $\phi^0_{s,t}$ and $\phi^1_{s,t}$ on $R_{s,t}$ by

$$\phi^0_{s,t} = \frac{\pi^2}{6}(-\log|t|)^{-2}(\triangle^{gr}_{s,t} - 2)^{-1}\Lambda \quad \text{and} \quad \phi^1_{s,t} = \phi_{s,t} - \phi^0_{s,t}. \tag{32}$$

Notice that in the above definition of $\phi^0_{s,t}$, we use the Laplacian with respect to g^{gr} (locally written as $\frac{1}{\omega}\frac{\partial^2}{\partial u \partial \bar{u}}$) instead of the Laplacian with respect to g^{hyp} (locally written as $\frac{1}{\rho}\frac{\partial^2}{\partial u \partial \bar{u}}$). This difference is controlled by (31). For simplicity, we will write ϕ, $\triangle$ instead of $\phi_{s,t}$, $\triangle^{gr}_{s,t}$ for the rest of the paper. In particular,

$$\phi = \frac{1}{2}\log\left(\frac{\omega}{\rho}\right), \qquad e^{2\phi} - 1 = O((-\log|t|)^{-2}). \tag{33}$$

Rewriting this another way,

$$\frac{\omega}{\rho} - 1 = O((-\log|t|)^{-2}) \text{ in } \mathcal{R}. \tag{34}$$

The function ϕ satisfies the standard curvature identity

$$\triangle\phi = e^{2\phi} + K^{gr} = (e^{2\phi} - 1) + (K^{gr} + 1)$$

and we can use this to obtain a gradient estimate. Indeed, estimates (33) and (20) imply

$$\triangle\phi = O((-\log|t|)^{-2}), \quad \phi = O((-\log|t|)^{-2}).$$

We now apply elliptic regularity and the Sobolev embedding theorem (cf. [Wo1], Estimate A3 in the Appendix). Here, an important point is that by lifting to the universal cover, we can obtain estimates that do not depend on the injectivity radius. *The key is that we have pointwise bounds on ϕ and its Laplacian.* In other words, fix $p > 2$ and radius equal to 1 to apply elliptic regularity in $B_1(0)$ for $x \in \mathcal{R}_{s,t}$. We have (for C independent of s, t)

$$||\phi||_{W^{1,p}(B_1(x))} \leq C\left(||\phi||_{L^p(B_1(x))} + ||\triangle\phi||_{L^p(B_1(0))}\right).$$

We then apply Sobolev embedding $W^{1,p} \subset C^0$, $p > 2$ to obtain

$$|\nabla\phi| = O((\log|t|)^{-2}) \tag{35}$$

where $\nabla = \nabla^{gr}_{s,t}$ is the gradient with respect to $g^{gr}_{s,t}$.

Rewriting (32), we have the functions $\phi^0, \phi^1 : \mathcal{R} \to (0, \infty)$ satisfying

$$\phi^0 = -\frac{\pi^2}{6}(-\log|t|)^{-2}(\triangle - 2)^{-1}\Lambda, \quad \phi^1 = \phi - \phi^0. \tag{36}$$

Thus,

$$\begin{aligned} \phi^0 &= O((-\log|t|)^{-2}), && (37)\\ \frac{\partial\phi^0}{\partial t} &= O(|t|^{-1}(-\log|t|)^{-3}) && (38)\\ \phi^1 &= O((-\log|t|)^{-4}). && (39) \end{aligned}$$

Lemma 22 *In $\mathcal{N}$, we have in coordinates (u, s, t) or (v, s, τ)*

$$\omega^{-\frac{1}{2}} \left| \frac{d\phi^0}{d\zeta} \right|, \ \omega^{-\frac{1}{2}} \left| \frac{d\phi^0}{d\bar{\zeta}} \right| = O((-\log|t|)^{-2}))$$

where $\zeta = u$ (resp. $\zeta = v$). Furthermore,

$$\omega^{-1} \left| \frac{d^2\phi^0}{d\zeta^2} \right|, \ \omega^{-1} \left| \frac{d^2\phi^0}{d\bar{\zeta}^2} \right|, \ \omega^{-1} \left| \frac{d^2\phi^0}{d\zeta d\bar{\zeta}} \right| = O((-\log|t|)^{-2})).$$

Proof. By (36) and (37),

$$\triangle\phi^0 = -\frac{\pi^2}{6}(-\log|t|)^{-2}\Lambda = O(\log|t|)^{-2}, \quad \phi^0 = O((-\log|t|)^{-2}). \tag{40}$$

By elliptic regularity and the Sobolev embedding theorem $W^{1,p} \subset C^0$, $p > 2$ (cf. proof of (35)), we have

$$|\nabla\phi^0|_{C^0} = O((-\log|t|)^{-2}).$$

Here, again, elliptic regularity is applied above with a ball of fixed radius for all t. This proves the first derivative estimates.

Additionally, the Kähler identities give

$$\triangle''(\partial\phi^0) \ = \ -\frac{1}{2}\partial\left((-\log|t|)^{-2}\Lambda\right),$$

and hence

$$\triangle''(\partial\phi^0) = O((-\log|t|)^{-2}), \quad |\partial\phi^0| = O((-\log|t|)^{-2}).$$

Applying elliptic regularity and the Sobolev embedding theorem as before,

$$|\nabla(\partial\phi^0)|_{C^0} = O((\log|t|)^{-2}).$$

Hence in $\mathcal{N}$

$$\omega^{-1} \left| \frac{d^2\phi^0}{du^2} \right| = O((\log|t|)^{-2}).$$

This implies the second derivative estimates. Q.E.D.

Lemma 23 *In $\mathcal{N}$ with respect to coordinates (u, s, t) or (v, s, τ)*

$$\omega^{-\frac{1}{2}}\left|\frac{d\phi^1}{d\zeta}\right|,\ \omega^{-\frac{1}{2}}\left|\frac{d\phi^1}{d\bar{\zeta}}\right| = O((-\log|t|)^{-4}))$$

where $\zeta = u$ (resp. $\zeta = v$). Furthermore,

$$\omega^{-1}\left|\frac{d^2\phi^1}{d\zeta^2}\right|,\ \omega^{-1}\left|\frac{d^2\phi^1}{d\bar{\zeta}^2}\right|,\ \omega^{-1}\left|\frac{d^2\phi^1}{d\zeta d\bar{\zeta}}\right| = O((-\log|t|)^{-2})).$$

Proof. By the standard identity,

$$\triangle\phi = e^{2\phi} + K^{gr},$$

we can write

$$\begin{aligned}
\triangle\phi^1 &= -\triangle\phi^0 + (e^{2\phi} - 1) + (K^{gr} + 1) \\
&= -(\triangle - 2)\phi^0 + (e^{2\phi} - 1 - 2\phi^0) + (K^{gr} + 1) \\
&= -\frac{\pi^2}{6}(\log|t|)^{-2}\Lambda + (e^{2\phi} - 1 - 2\phi^0) + (K^{gr} + 1).
\end{aligned}$$

By [Wo1] page 445, the curvature formula in proof of Expansion 4.2,

$$-\frac{\pi^2}{6}(\log|t|)^{-2}\Lambda + K^{gr} + 1 = O((-\log|t|)^{-4}), \quad \phi^1 = O((-\log|t|)^{-4}.$$

Thus,

$$\triangle\phi^1 = O(-\log|t|)^{-4}, \quad \phi^1 = O((-\log|t|)^{-4}). \tag{41}$$

By elliptic regularity as before,

$$|\nabla\phi^1|_{C^0} \leq C\left(|\phi^1|_{C^0} + |\triangle\phi^1|_{C^0}\right) = O((\log|t|)^{-4}).$$

and hence also

$$|\partial\phi^1| = O((\log|t|)^{-4}).$$

The proof of the second estimates is the same as that of ϕ^0. Q.E.D.

Remark 24 The second derivatives of ϕ^1 have better estimates, but we will need them in this paper.

Lemma 25 *In $\mathcal{N}$ with respect to coordinates (u, s, t) or (v, s, τ)*

$$\omega^{-\frac{1}{2}} \left| \frac{d\phi}{d\zeta} \right|, \ \omega^{-\frac{1}{2}} \left| \frac{d\phi}{d\bar{\zeta}} \right| = O((-\log |t|)^{-2})) \tag{42}$$

where $\zeta = u$ (resp. $\zeta = v$). Furthermore,

$$\omega^{-1} \left| \frac{d^2\phi}{d\zeta^2} \right|, \ \omega^{-1} \left| \frac{d^2\phi}{d\bar{\zeta}^2} \right|, \ \omega^{-1} \left| \frac{d^2\phi}{d\zeta d\bar{\zeta}} \right| = O((-\log |t|)^{-2})). \tag{43}$$

Proof. Combine proof of Lemma 22 and Lemma 23. q.e.d.

We now define a vector field W on the total space $\mathcal{R}$. For the coordinates (v, s, t) in $\mathcal{N}$, we will set $\tau = t$ in order to distinguish them from the coordinates (u, s, t) in $\mathcal{N}$. Consider the coordinate vector fields

$$\frac{\partial}{\partial t} \text{ defined in } \mathcal{N} \text{ with respect to the coordinates } (u, s, t)$$

and

$$\frac{\partial}{\partial \tau} \text{ defined in } \mathcal{N} \text{ with respect to the coordinates } (v, s, \tau). \tag{44}$$

Remark 26 The reason we use the variable τ in expressing the coordinates (v, s, τ) is to distinguish the above two vector fields. Note that if a point $p_0 \in \mathcal{N}$ is given by (u_0, s_0, t_0) in the coordinates (u, s, t) and (v_0, s_0, τ_0) in the coordinates (v, s, τ), then $t_0 = \tau_0$. On the other hand, as it will be explained below (cf. (45)), $p \mapsto \frac{\partial}{\partial t}(p)$ and $p \mapsto \frac{\partial}{\partial \tau}(p)$ for $p \in \mathcal{N}$ are two distinct vector fields.

Using the identification $f_s(u) g_s(v) = t = \tau$, we write

$$u(v, \tau) = f_s^{-1}(\frac{\tau}{g_s(v)}), \quad \frac{\partial u}{\partial \tau}(v, \tau) = \frac{\partial}{\partial \tau} \left(f_s^{-1}(\frac{\tau}{g_s(v)}) \right).$$

We apply the change of the coordinates $u(v, \tau)$ to obtain the expression of $\frac{\partial}{\partial \tau}$ in terms of the coordinates (u, t, s) as

$$\frac{\partial}{\partial \tau} = \frac{\partial t}{\partial \tau} \frac{\partial}{\partial t} + \frac{\partial u}{\partial \tau} \frac{\partial}{\partial u} = \frac{\partial}{\partial t} + H(u, t) \frac{\partial}{\partial u} \tag{45}$$

where $H(u,t)$ is a function (holomorphic in u) defined by

$$H(u,t) := \frac{\partial u}{\partial \tau}\Big|_{v=g_s^{-1}(\frac{t}{f_s(u)}),\ \tau=t}. \tag{46}$$

Thus, $H(u,t)\frac{\partial}{\partial u}$ is the expression with respect to coordinates (u,s,t) of the vector field $\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau}$ defined in $\mathcal{N}$.

We now use the function H to define a C^∞ vector field W in $\mathcal{N}$ by setting

$$W = \begin{cases} \dfrac{\partial}{\partial t} + \eta H \dfrac{\partial}{\partial u} & \text{in } \mathcal{N}^1 \text{ with respect to } (u,s,t) \\ \dfrac{\partial}{\partial \tau} & \text{in } \mathcal{N}^2 \backslash \mathcal{N}^1 \text{ with respect to } (v,s,\tau=t) \end{cases} \tag{47}$$

where η is a rotationally symmetric bump function such that $\eta \equiv 0$ for $|u| \geq |t|^{\frac{1}{2}-\delta}$ and $\eta \equiv 1$ for $|u| \leq |t|^{\frac{1}{2}+\delta}$ as in [Wo1] 3.4 CG. In other words, the vector field W in $\mathcal{N}^1 \cap \mathcal{N}^2$ is the interpolation of the vector field $\frac{\partial}{\partial t}$ defined in $\mathcal{N}^1$ with respect to the local coordinates (u,s,t) and the vector field $\frac{\partial}{\partial \tau}$ defined in $\mathcal{N}^2$ with respect to the local coordinates $(v,s,\tau=t)$. The vector field W is C^∞ in $\mathcal{N}$. We can extend W as a C^∞ vector field on the total space $\mathcal{R}$. Indeed, $\mathcal{R}$ is trivial away from the pinching region $\mathcal{N}^1 \cap \mathcal{N}^2$ and the product structure defines a canonical lift of the the vector field $\frac{\partial}{\partial t}$ defined $\mathcal{S}$.

Definition 27 We denote by W the C^∞ vector field on $\mathcal{R}$ defined by (47) in $\mathcal{N}$ and the canonical lift to $\mathcal{R}\backslash\mathcal{N}$ of the vector field $\frac{\partial}{\partial t}$ defined $\mathcal{S}$.

We will now derive some estimates involving the function H. First observe that if $f_s(u)$ and $g_s(v)$ are the identity maps, then $H(u,t) = \frac{u}{t}$. Since $f_s(u)$ and $g_s(v)$ are holomorphic functions close to the identity,

$$H = O(|t|^{-1})(u + O(|u|^2)). \tag{48}$$

We will record some other estimates involving the function $H(u,t)$ below.

Lemma 28 *In $\mathcal{N}^1 \cap \mathcal{N}^2$, we have with respect to the coordinates (u,s,t),*

$$H = \frac{u}{t} + O(|t|^{-2\delta}) \tag{49}$$

and

$$Hh^{\frac{1}{2}} = O(|t|^{-1}(-\log|t|)^{-1}).$$

Proof. The first assertion follows immediately from (48). Since $h = O(|u|^{-2}(-\log|t|)^{-2})$ in $\mathcal{N}^1 \cap \mathcal{N}^2$, the second assertion follows from the first. Q.E.D.

Lemma 29 *In $\mathcal{N}$,*

$$WK^{gr} = O(|t|^{-1}(-\log|t|)^{-3}))$$

where W is given in Definition 27.

Proof. In $\mathcal{N}^1 \backslash \mathcal{N}^2$, the estimate follows from Lemma 20 since $W = \frac{\partial}{\partial t}$. In $\mathcal{N}^1 \cap \mathcal{N}^2$ with respect to coordinates (u,s,t),

$$WK^{gr} = \frac{\partial K^{gr}}{\partial t} + \eta H \frac{\partial K^{gr}}{\partial u}.$$

The first term on the right hand side is $O(|t|^{-1}(-\log|t|)^{-3}))$ by Lemma 20. The second term is $O(|t|^{-\frac{1}{2}-24\delta})$ by Lemma 21 and Lemma 28. This proves the estimate in $\mathcal{N}^1$. Similar argument proves the estimate in $\mathcal{N}^2$. Q.E.D.

Lemma 30 *In $\mathcal{N}$ with respect to the coordinates (u,s,t),*

$$\begin{aligned}
\frac{Wh}{h} &= O(|t|^{-1}(-\log|t|)^{-3})(-\log|u|)^2 \\
W\left(\frac{h}{\omega}\right) &= O(|t|^{-1}(-\log|t|)^{-3}) \\
\frac{W\omega}{\omega} &= O(|t|^{-1}(-\log|t|)^{-3})(-\log|u|)^2
\end{aligned}$$

where W is given in Definition 27.

Proof. In $\mathcal{N}^1 \backslash \mathcal{N}^2$, the first estimate follows from Lemma 10. In $\mathcal{N}^1 \cap \mathcal{N}^2$, we have by Lemma 10 and Lemma 28 that

$$\begin{aligned}
\frac{Wh}{h} &= \frac{\frac{\partial h}{\partial t}}{h} + \eta H \frac{\frac{\partial h}{\partial u}}{h} \\
&= O(|t|^{-1}(-\log|t|)^{-3})(-\log|u|)^2 + \frac{u}{t} O(|u|^{-1}(-\log|u|)^{-1}) \\
&= O(|t|^{-1}(-\log|t|)^{-3})(-\log|u|)^2 + O(|t|^{-1}(-\log|u|)^{-1}).
\end{aligned}$$

Thus, the first estimate in $\mathcal{N}^1 \cap \mathcal{N}^2$ follows from the fact that $(-\log|u|) \leq (\frac{1}{2} + 2\delta)(-\log|t|)$. The second estimate follows from Lemma 18, Lemma 19 and Lemma 28. The third estimate follows from combining the first and the second estimates. Q.E.D.

Lemma 31 *In $\mathcal{R}$,*

$$W\left(\frac{\omega}{\rho}\right) = O(|t|^{-1}(-\log|t|)^{-2})$$

where W is given in Definition 27.

Proof. We differentiate the standard identity

$$\triangle\phi = e^{2\phi} + K^{gr} \tag{50}$$

with respect to W to obtain

$$(\triangle - 2e^{2\phi})W\phi \quad = \quad WK^{gr} + \text{comm}$$

where the term comm comes from commuting W and $\triangle$. Using the fact that H is holomorphic in u (cf. (46)), we obtain

$$\begin{aligned}\text{comm} &= \triangle W\ \phi - W\ \triangle\phi \\ &= \frac{\partial}{\partial\zeta}(\eta H)\triangle\phi + \frac{\partial}{\partial\bar{\zeta}}(\eta H)\frac{1}{\omega}\frac{\partial^2\phi}{\partial\zeta^2} + \triangle(\eta H)\frac{\partial\phi}{\partial\zeta} + \eta H\frac{\frac{\partial\omega}{\partial\zeta}}{\omega}\triangle\phi \\ &= \eta\frac{\partial H}{\partial\zeta}\triangle\phi + H\frac{\partial\eta}{\partial\zeta}\triangle\phi + H\frac{\partial\eta}{\partial\bar{\zeta}}\frac{1}{\omega}\frac{\partial^2\phi}{\partial\zeta^2} + H\triangle\eta\frac{\partial\phi}{\partial\zeta} \\ &\quad + \frac{1}{\omega}\frac{\partial\eta}{\partial\bar{\zeta}}\frac{\partial H}{\partial\zeta}\frac{\partial\phi}{\partial\zeta} + \eta H\frac{\frac{\partial\omega}{\partial\zeta}}{\omega}\triangle\phi.\end{aligned}$$

(i) Since $H = O(|t|^{-1})(u + O(|u|^2))$ (cf. 48) and H is holomorphic (cf. (46)), $\frac{\partial H}{\partial\zeta} = O(|t|^{-1})(1 + O(|u|)$. Moreover by (40), $\triangle\phi = O((-\log|t|)^{-2})$. It follows that the first term is $O(|t|^{-1}(-\log|t|)^{-2})$.
(ii) By Lemma 16, $\frac{\partial\eta}{\partial\zeta}\ \frac{\partial\eta}{\partial\zeta} = O|u|^{-1}((-\log|t|)^{-1})$, hence combining with the estimates used before we obtain that the second term is $O(|t|^{-1}(-\log|t|)^{-3})$.

(iii) By Lemma 25, $\frac{1}{\omega}\frac{\partial^2\phi}{\partial\zeta^2} = O((-\log|t|)^{-2}))$, hence combining with the estimates used before we obtain that the third term is $O(|t|^{-1}(-\log|t|)^{-3})$.
(iv) By Lemma 17 and Lemma 18, $\triangle\eta = O(1)$. By Lemma 25, $\omega^{-\frac{1}{2}}\frac{\partial\phi}{\partial\zeta} = O((-\log|t|)^{-2})$ and $\omega^{\frac{1}{2}} = O(|u|^{-1}(-\log|u|)^{-1})$, hence $\frac{\partial\phi}{\partial\zeta} = O(|u|^{-1}(-\log|t|)^{-2})$. Hence, combining with the estimate for H used before we obtain that the forth term is $O(|t|^{-1}(-\log|t|)^{-2})$.
(v) By Lemma 17 and Lemma 18, $\omega^{-\frac{1}{2}}\frac{\partial\eta}{\partial\zeta} = O(1)$ and by Lemma 25, $\omega^{-\frac{1}{2}}\frac{\partial\phi}{\partial\zeta} = O((-\log|t|)^{-2})$. Combined with $\frac{\partial H}{\partial\zeta} = O(|t|^{-1})$, we obtain that the fifth term is $O(|t|^{-1}(-\log|t|)^{-2})$.
(vi) By Lemma 10 and Lemma 20, $\frac{\frac{\partial\omega}{\partial\zeta}}{\omega} = O(|u|^{-1}(-\log|u|)^{-1})$. Combined with estimated used before, we obtain that the sixth term is $O(|t|^{-1}(-\log|t|)^{-3})$.

In summary, we have shown

$$(\triangle\phi - 2e^{2\phi})W\phi = O(|t|^{-1}(-\log|t|)^{-2})$$

in either $\mathcal{N}^1$ with $\zeta = u$ or in $\mathcal{N}^2$ with $\zeta = v$. Since the C^k-estimates of ϕ are uniformly bounded independent of t and s outside of the pinching region $\mathcal{N}$, (51) implies

$$(\triangle - 2e^{2\phi})W\phi = O(|t|^{-1}(-\log|t|)^{-2}) \text{ in } \mathcal{R},$$

and hence by the maximum principle (cf. [Wo1] Appendix A.2)

$$W\phi = O(|t|^{-1}(-\log|t|)^{-2}). \tag{51}$$

Q.E.D.

Lemma 32 *In $\mathcal{N}^1$ (resp. $\mathcal{N}^2$) with respect to the coordinates (u,s,t) (resp. (v,s,t)),*

$$\frac{\partial}{\partial t}\left(\frac{\omega}{\rho}\right) = O(|t|^{-1}(-\log|t|)^{-3}).$$

Proof. From (36), we have

$$(\triangle - 2)\phi^0 = -\frac{\pi^2\Lambda}{6}(-\log|t|)^{-2}).$$

We differentiate this with respect to W to obtain

$$W(\triangle - 2)\phi^0 = O(|t|^{-1}(-\log|t|)^{-3}). \tag{52}$$

From (50), we obtain

$$W(\triangle - 2)\phi^1 = -W(\triangle - 2)\phi^0 + WK^{gr}$$

or by commuting W with $\triangle$

$$(\triangle - 2)W\phi^1 = -W(\triangle - 2)\phi^0 + WK^{gr} + comm.$$

We now estimate the terms in the right hand side above. By (52), the first term is $O(|t|^{-1}(-\log|t|)^{-3})$ and by Lemma 29, the second term is $O(|t|^{-1}(-\log|t|)^{-3})$. Arguing analogously as in the proof of Lemma 31 and noting that we have a better estimates for ϕ^1 than for ϕ (compare Lemma 23 with Lemma 25), we obtain

$$\text{comm} = O(|t|^{-1}(-\log|t|)^{-3}).$$

Indeed, in (i) of the proof of Lemma 31, by (41), $\triangle\phi^1 = (-\log|t|)^{-4}$, hence the first term is $O(|t|^{-1}(-\log|t|)^{-4})$. In (ii) and (iii) we have the desired estimate. Furthermore, in (iv) and (v), we have by Lemma 23 that $\omega^{-\frac{1}{2}}\frac{\partial\phi^1}{\partial\zeta} = O((-\log|t|)^{-4})$ instead of $O((-\log|t|)^{-2})$ for $\omega^{-\frac{1}{2}}\frac{\partial\phi^0}{\partial\zeta}$ and this accounts for the extra $(-\log|t|)^{-1}$.

Thus, the maximum principle as before implies

$$W\phi^1 = O(|t|^{-1}(-\log|t|)^{-3}).$$

Lemma 18, Lemma 23 and Lemma 28, imply that

$$\eta H\frac{\partial\phi^1}{\partial u} = \eta H h^{\frac{1}{2}}h^{-\frac{1}{2}}\frac{\partial\phi^1}{\partial u} = O(|t|^{-1}(-\log|t|)^{-5}) \text{ in } \mathcal{N}_1$$

which in turn implies that

$$\frac{\partial\phi^1}{\partial t} = W\phi^1 - \eta H\frac{\partial\phi^1}{\partial u} = O(|t|^{-1}(-\log|t|)^{-3}) \text{ in } \mathcal{N}_1.$$

Combined with (38), we conclude

$$\frac{\partial\phi}{\partial t} = O(|t|^{-1}(-\log|t|)^{-3}) \text{ in } \mathcal{N}_1$$

and similarly in $\mathcal{N}_2$. Q.E.D.

5 The proof of the main results

Let R be a nodal surface with nodes $\{n_i\}_{i=1}^N$ and $R_0 = R\backslash\{n_i\}_{i=1}^N$ be the Riemann surface with punctures $\{a_i, b_i\}$ associated with each node n_i. For $s \in B_\epsilon(0) \subset \mathbf{C}^n$, let R_s be the family of Riemann surfaces defined by the Beltrami differentials as in (2). The resolution of the singularities (described in Section 2 for a single node) can be simultaneously carried out for all R_s and nodes, and we obtain a family of Riemann surfaces

$$F : \mathcal{R} \to \mathcal{S} \subset \mathbf{C}^n \times \mathbf{C}^N$$

whose fiber over $(s,t) = (s, t_1, \ldots, t_N) \in \mathbf{C}^n \times \mathbf{C}^N$ is the Riemann surface $R_{s,t}$. As in the case of one node, the coordinates (s,t) are called the *plumbing coordinates* and the Riemann surface $R_{s,t}$ is the surface corresponding to the coordinates (s,t) via the *plumbing construction*.

For each $i = 1, \ldots, N$, we denote by $\mathcal{N}^i$, $\mathcal{N}^{i,1}$ and $\mathcal{N}^{i,2}$ the neighborhoods defined by the plumbing construction for the node n_i as defined by (5) with t_i replacing t. Let (u_i, s, t) and $(v_i, s, \tau = t)$ be the local coordinates of $\mathcal{N}^i$. Fix s and $t = (t^1, \ldots, t^N)$ and define

$$N_{s,t}^{i,+} := \{u_i \in \mathcal{N}^{i,1} \cap R_{s,t} : |f_s(u_i)| \geq |t_i|^{\frac{1}{2}}\}$$

and

$$N_{s,t}^{i,-} := \{v_i \in \mathcal{N}^{i,2} \cap R_{s,t} : |g_s(v_i)| \geq |t_i|^{\frac{1}{2}}\}.$$

By (5),

$$\mathcal{N}_{s,t}^i = \mathcal{N}^i \cap R_{s,t} = N_{s,t}^{i,+} \cup N_{s,t}^{i,-}$$

and

$$\begin{aligned} N_{s,t}^{i,+} \cap N_{s,t}^{i,-} &= \{u_i \in \mathcal{N}^{i,1} \cap R_{s,t} : |f_s(u_i)| = |t_i|^{\frac{1}{2}}\} \\ &= \{v_i \in \mathcal{N}^{i,2} \cap R_{s,t} : |g_s(v_i)| = |t_i|^{\frac{1}{2}}\}. \end{aligned}$$

Fix i and let W_i be the vector field given as W in Definition 27 with $\mathcal{N}^i$ playing the role of $\mathcal{N}$. In other words, W_i is $\frac{\partial}{\partial t_i}$ in $\mathcal{R}_{s,t}\backslash(\mathcal{N}^{i,1} \cap \mathcal{N}^{i,2})$ and defined by interpolation in $\mathcal{N}^{i,1} \cap \mathcal{N}^{2,i}$. In $N_{s,t}^{i,+}$ with respect to the coordinate u_i, we have

$$W_i(u_i) = \frac{\partial}{\partial t_i} + \eta H_i \frac{\partial}{\partial u_i} \quad \text{where} \quad H_i(u_i) := \frac{\partial u_i}{\partial t_i}\Big|_{v_i = g_s^{-1}(\frac{t_i}{f_s(u_i)})}.$$

In $N_{s,t}^{i,-}$ with respect to the coordinate v_i (with $\tau_i = t_i$),

$$W_i(v_i) = \frac{\partial}{\partial \tau_i} + \tilde{\eta}\tilde{H}_i \frac{\partial}{\partial v_i} \quad \text{where} \quad \tilde{H}_i(v_i) := \frac{\partial v_i}{\partial \tau_i}\Big|_{u_i = f_s^{-1}(\frac{\tau_i}{g_s(u_i)})}.$$

Here, $\eta \equiv 0$, $\tilde{\eta} \equiv 1$ for $|u_i| \geq |t_i|^{\frac{1}{2}-\delta}$ and $\tilde{\eta} \equiv 0$, $\eta \equiv 1$ for $|v_i| \geq |\tau_i|^{\frac{1}{2}-\delta}$. Applying change of variables, we obtain

$$\frac{\partial}{\partial \tau_i} + \tilde{\eta}\tilde{H}_i \frac{\partial}{\partial v_i} = \frac{\partial}{\partial t_i} + \left(\frac{\partial u_i}{\partial t_i} + \tilde{\eta}\frac{\partial v_i}{\partial \tau_i}\frac{\partial u_i}{\partial v_i}\right)\frac{\partial}{\partial u_i}.$$

Thus,

$$\frac{\partial u_i}{\partial t_i} + \tilde{\eta}\frac{\partial v_i}{\partial \tau_i}\frac{\partial u_i}{\partial v_i} = \eta\frac{\partial u_i}{\partial t_i}.$$

With this, we can derive a relationship between the two functions η and $\tilde{\eta}$. Differentiating $f_s(u_i)g_s(v_i) = t_i = \tau_i$, we have

$$\frac{\partial u_i}{\partial t_i}f_s'(u_i) = \frac{1}{g_s(v_i)}, \quad \frac{\partial v_i}{\partial \tau_i}g_s'(u) = \frac{1}{f_s(u_i)} = \frac{g_s(v_i)}{t_i}, \quad \frac{\partial u_i}{\partial v_i}f_s'(u_i) = -\frac{t_i g_s'(v_i)}{g_s^2(v_i)},$$

and hence

$$\frac{1}{g_s(v_i)f_s'(u_i)} - \tilde{\eta}\frac{1}{g_s(v_i)} = \eta\frac{1}{g_s(v_i)f_s'(u_i)}$$

which in turn implies

$$\eta + \frac{1}{f_s'(u_i)}\tilde{\eta} = \frac{1}{f_s'(u_i)}.$$

For convenience, we choose η such that

$$\eta(u_i) = \frac{1}{2} \text{ for } |f_s(u_i)| = |t_i|^{\frac{1}{2}}. \tag{53}$$

Noting that $f_s'(u_i) = 1 + O(|u_i|)$ (cf. (3) and (4)), we then obtain

$$\tilde{\eta}(v_i) = \frac{1}{2} + O(|t_i|^{\frac{1}{2}}) \text{ for } |g_s(v_i)| = |t_i|^{\frac{1}{2}}. \tag{54}$$

We now recall the Masur differentials $\{\Phi_\mu\}$ (cf. [Mas]) corresponding to cotangent vectors in a neighborhood at a point at the boundary of Teichmüller space. For clarity, we will use the lower case i for the Masur differential corresponding dual of the tangent vector $\frac{\partial}{\partial t_i}$ of the boundary of Teichmüller space and the upper case I for the Masur differential corresponding to the tangent vector $\frac{\partial}{\partial s_I}$ normal to the boundary. We can express these differentials in the l^{th} neighborhood $\mathcal{N}^l$ with respect to the coordinates (u_l, s, t) as

$$\Phi_i = \phi_i dz^2 \quad \text{and} \quad \Phi_I = \phi_I dz^2$$

where

$$\phi_i(u_l, s, t) = -\frac{t_i}{\pi}\left(\frac{\delta_{il}}{u_l^2} + a_{-1}(u_l, s, t) + \frac{1}{u_l^2}\sum_{j=1}^{\infty}\left(\frac{t_l}{u_l}\right)^2 \cdot t_l^{m(j)} \cdot a_j(s,t)\right) \tag{55}$$

with $m(j) \geq 0$, a_{-1} with at most a simple pole at 0, a_j, $j \geq 1$ are holomorphic at 0 and

$$\phi_I(u_l, s, t) = \phi_I(u_l, 0, 0) + \frac{1}{u_l^2}\sum_{j=1}^{\infty}\left(\frac{t_l}{u_l}\right)^j \cdot t_l^{\tilde{m}(j)} \cdot b_j(s,t) + \sum_{j=-1}^{\infty} u_l^j \cdot c_j(s,t) \tag{56}$$

with $\tilde{m}(l) \geq 0$, $\phi_I(u_l, 0, 0)$ has at most a simple pole and b_l, c_l are holomorphic at 0. Since $\frac{|t_l|}{|u_l|^2} \leq 1$ in $N_{s,t}^{l,+}$, we have

$$\begin{aligned}
\phi_i = O(\frac{|t_i|}{|u_i|^2}), &\qquad \frac{\partial \phi_i}{\partial t_i} = O(\frac{1}{|u_i|^2}), \text{ on } N_{s,t}^{i,+} \\
\phi_i = O\left(\frac{|t_i|}{|u_l|}\right), &\qquad \frac{\partial \phi_i}{\partial t_i} = O(\frac{1}{|u_l|}) \text{ on } N_{s,t}^{l,+} \\
\phi_i = O(|t_i|), &\qquad \frac{\partial \phi_i}{\partial t_i} = O(1) \text{ everywhere else.}
\end{aligned} \tag{57}$$

Moreover,

$$\begin{aligned}
\phi_I = O(\frac{1}{|u_i|}), &\qquad \frac{\partial \phi_I}{\partial t_i} = O(\frac{1}{|u_i|^3}) \text{ on } N_{s,t}^{i,+} \\
\phi_I = O(1), &\qquad \frac{\partial \phi_I}{\partial t_i} = O(\frac{1}{|u_l|}) \text{ on } N_{s,t}^{l,+} \\
\phi_I = O(1) &\qquad \frac{\partial \phi_I}{\partial t_i} = O(1) \text{ everywhere else.}
\end{aligned} \tag{58}$$

We start with following simple results.

Lemma 33 *With* $h(z)=\left(\frac{\pi}{\log|t|}\csc(\frac{\pi\log|z|}{\log|t|})\frac{|dz|}{|z|}\right)^2$, *we have*

$$\int_{\{|t|^{\frac{1}{2}}\le|z|\le1\}}\frac{1}{|z|^4}\frac{1}{h(z)}dxdy=\frac{(-\log|t|)^3}{2\pi}$$

and

$$\int_{\{|t|^{\frac{1}{2}}\le|z|\le1\}}\frac{1}{|z|^n}\frac{1}{h(z)}dxdy=O(1)\quad \textit{for } n<4.$$

Furthermore,

$$\int_{\{|t|^{\frac{1}{2}}\le|z|\le1\}}\frac{1}{|z|^4}\frac{1}{h(z)}\frac{\frac{\partial h}{\partial t}(z)}{h(z)}dxdy=\frac{(-\log|t|)^2}{4\pi t}.$$

Proof. A straight forward computation using the substitution $\theta=\frac{\pi\log r}{\log|t|}$, $d\theta=\frac{\pi}{r\log|t|}dr$, $r=e^{\frac{\theta\log|t|}{\pi}}=|t|^{\frac{\theta}{\pi}}$ yields

$$\begin{aligned}
&\int_{\{|t|^{\frac{1}{2}}\le|z|\le1\}}\frac{1}{|z|^n}\frac{1}{h(z)}dxdy\\
=&\ \frac{(-\log|t|)^2}{\pi^2}\int_{\{|t|^{\frac{1}{2}}\le|z|\le1\}}\frac{1}{|z|^{n-2}}\sin^2(\frac{\pi\log|z|}{\log|t|})dxdy\\
=&\ \frac{(-\log|t|)^2}{\pi^2}\int_{\{|t|^{\frac{1}{2}}\le r\le1\}}\frac{1}{r^{n-2}}\sin^2(\frac{\pi\log r}{\log|t|})2\pi rdr\\
=&\ \frac{2(-\log|t|)^3}{\pi^2}\int_0^{\frac{\pi}{2}}e^{(4-n)a\theta}\sin^2\theta d\theta\qquad(\text{where } a=\frac{\log|t|}{\pi})\\
=&\ \frac{(-\log|t|)^3}{\pi^2}\int_0^{\frac{\pi}{2}}e^{(4-n)a\theta}(1-\cos2\theta)d\theta.
\end{aligned}$$

Letting $n=4$, we have

$$\begin{aligned}
\int_{\{|t|^{\frac{1}{2}}\le|z|\le1\}}\frac{1}{|z|^4}\frac{1}{h(z)}dxdy &= \frac{(-\log|t|)^3}{\pi^2}\int_0^{\frac{\pi}{2}}1-\cos2\theta d\theta\\
&= \frac{(-\log|t|)^3}{2\pi}
\end{aligned}$$

which is the first estimate. Applying the integral formula

$$\begin{aligned}
\int e^{(4-n)a\theta}\cos2\theta d\theta &= \frac{e^{(4-n)a\theta}((4-n)\cos2\theta+2\sin2\theta)}{(4-n)^2a^2+4}+K,\quad a=\frac{\log|t|}{\pi}\\
&= \frac{r^{4-n}((4-n)\cos2\theta+2\sin2\theta)}{(4-n)^2a^2+4}+K
\end{aligned}$$

the second estimate follows for $n < 4$. By Lemma 10, we write

$$\int_{\{|t|^{\frac{1}{2}}\leq|z|\leq 1\}} \frac{1}{|z|^n}\frac{1}{h(z)}\frac{\frac{\partial h}{\partial t}(z)}{h(z)}dxdy \quad = \quad \frac{1}{t(-\log|t|)}\int_{\{|t|^{\frac{1}{2}}\leq|z|\leq 1\}} \frac{1}{|z|^n}\frac{1}{h(z)}(1-\theta\cot\theta)du^1du^2.$$

Using the same substitution as above, we obtain

$$\int_{\{|t|^{\frac{1}{2}}\leq|z|\leq 1\}} \frac{1}{|z|^n}\frac{1}{h(z)}\theta\cot\theta dxdy \quad = \quad \frac{2(-\log|t|)^3}{\pi^2}\int_0^{\frac{\pi}{2}} e^{(4-n)a\theta}\theta\sin\theta\cos\theta \ d\theta.$$

Letting $n = 4$ in the above equality, we obtain

$$\begin{aligned}\int_{\{|t|^{\frac{1}{2}}\leq|z|\leq 1\}} \frac{1}{|z|^4}\frac{1}{h(z)}\theta\cot\theta dxdy \quad &= \quad \frac{(-\log|t|)^3}{\pi^2}\int_0^{\frac{\pi}{2}} \theta\sin 2\theta \ d\theta \\ &= \quad \frac{(-\log|t|)^3}{4\pi}.\end{aligned}$$

Combining this with the first estimate proves the third estimate. Q.E.D.

Lemma 34 *If $\{\Phi_\mu = \phi_\mu dz^2\}$ is the set of Masur differentials, $g_{s,t}$ a metric along the fibers of $\mathcal{R}$ and $\mathcal{G} = \frac{\Phi_\mu\bar{\Phi}_\mu}{g^{gr}_{s,t}}$, then*

$$\begin{aligned}\frac{\partial}{\partial t_i}\int_{R_{s,t}}\mathcal{G} \quad &= \quad \int_{N^{i,+}_{s,t}}\frac{\partial}{\partial t_i}\mathcal{G} + \int_{N^{i,-}_{s,t}}\frac{\partial}{\partial t_i}\mathcal{G} + \int_{R_{s,t}\setminus\mathcal{N}^i_{s,t}}\frac{\partial}{\partial t_i}\mathcal{G} \\ &+ \quad \int_{B^{i,+}_{s,t}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\mathcal{G} + \int_{B^{i,-}_{s,t}} i_{\tilde{\eta}\tilde{H}_i(v_i)\frac{\partial}{\partial v_i}}\mathcal{G}.\end{aligned}$$

In the above formula i denotes interior multiplication and $B^{i,\pm}_{s,t}$ denotes the inner boundary circle of $N^{i,\pm}_{s,t}$ with the induced orientation.

Proof. By definition of W_i, the projection F_*W_i to $\mathcal{S}$ is equal to $\frac{\partial}{\partial t_i}$. Thus,

$$\frac{\partial}{\partial t_i}\int_{R_{s,t}}\mathcal{G} = \int_{R_{s,t}}\mathcal{L}_{W_i}\mathcal{G}.$$

Since $W_i = \frac{\partial}{\partial t_i} + \eta H_i\frac{\partial}{\partial t_i}$ in $N^{i,+}_{s,t}$, $W_i = \frac{\partial}{\partial t_i} + \tilde{\eta}\tilde{H}_i$ in $N^{i,-}_{s,t}$ and $\frac{\partial}{\partial t_i}$ in $R_{s,t}\backslash calN^i_{s,t}$, it suffices to prove

$$\int_{N^{i,+}_{s,t}}\mathcal{L}_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\mathcal{G} = \int_{B^{i,+}_{s,t}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\mathcal{G}$$

and similarly for $N^{i,-}_{s,t}$. Indeed, this follows by Stokes theorem

$$\int_{N^{i,+}_{s,t}} \mathcal{L}_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\mathcal{G} = \int_{N^{i,+}_{s,t}} d i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\mathcal{G} = \int_{B^{i,+}_{s,t}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\mathcal{G}.$$

Q.E.D.

Lemma 35 *If Φ_i is the Masur differential dual to $\frac{\partial}{\partial t_i}$, then*

$$\frac{\partial}{\partial t_i}\int_{R_{s,t}}\left(\frac{\Phi_i\bar{\Phi}_i}{g^{hyp}_{s,t}} - \frac{\Phi_i\bar{\Phi}_i}{g^{gr}_{s,t}}\right) \;=\; O(|t_i|(-\log|t_i|)).$$

If Φ_I and Φ_J are the Masur differentials dual to $\frac{\partial}{\partial t_I}$ and $\frac{\partial}{\partial t_J}$, then

$$\frac{\partial}{\partial t_i}\int_{R_{s,t}}\left(\frac{\Phi_I\bar{\Phi}_J}{g^{hyp}_{s,t}} - \frac{\Phi_J\bar{\Phi}_i}{g^{gr}_{s,t}}\right) \;=\; O(|t_i|^{-1}(-\log|t_i|)^{-3}).$$

Proof. We can write

$$\begin{aligned}
\int_{N^{i,+}_{s,t}}\frac{\partial}{\partial t_i}\left(\frac{\phi_i\bar{\phi}_i}{\rho} - \frac{\phi_i\bar{\phi}_i}{\omega}\right)du_i^1du_i^2 &= \int_{N^{i,+}_{s,t}}\frac{\partial}{\partial t_i}\left(\frac{\phi_i\bar{\phi}_i}{\omega}\left(\frac{\omega}{\rho}-1\right)\right)du_i^1du_i^2 \\
= \int_{N^{i,+}_{s,t}}\frac{\frac{\partial\phi_i}{\partial t_i}\bar{\phi}_i}{\omega}\left(\frac{\omega}{\rho}-1\right)du_i^1du_i^2 &+ \int_{N^{i,+}_{s,t}}\frac{\phi_i\bar{\phi}_i}{\omega}\left(-\frac{\frac{\partial\omega}{\partial t_i}}{\omega}\right)\left(\frac{\omega}{\rho}-1\right)\;du_i^1du_i^2 \\
+\int_{N^{i,+}_{s,t}}\frac{\phi_i\bar{\phi}_i}{\omega}\frac{\partial}{\partial t_i}\left(\frac{\omega}{\rho}-1\right)\;du_i^1du_i^2. & \qquad (59)
\end{aligned}$$

By (34), (57), Lemma 18, Lemma 32 and Lemma 33,

$$\begin{aligned}
\int_{N^{i,+}_{s,t}}\frac{\frac{\partial\phi_i}{\partial t_i}\bar{\phi}_i}{h}\frac{h}{\omega}\left(\frac{\omega}{\rho}-1\right)\;du_i^1du_i^2 &= O(|t_i|(-\log|t_i|)^{-2})\int_{N^{i,+}_{s,t}}\frac{1}{|u_i|^4}\frac{1}{h}du_i^1du_i^2 \\
&= O(|t_i|(-\log|t_i|)), \qquad (60)
\end{aligned}$$

$$\begin{aligned}
&\int_{N^{i,+}_{s,t}}\frac{\phi_i\bar{\phi}_i}{h}\frac{h}{\omega}\left(-\frac{\frac{\partial\omega}{\partial t_i}}{\omega}\right)\left(\frac{\omega}{\rho}-1\right)\;du_i^1du_i^2 \\
&= O(|t_i|(-\log|t_i|)^{-3})\int_{N^{i,+}_{s,t}}\frac{1}{|u_i|^4}\frac{1}{h}\;du_i^1du_i^2 \\
&= O(|t_i|) \qquad (61)
\end{aligned}$$

and

$$\int_{N^{i,+}_{s,t}} \frac{\phi_i\bar{\phi}_i}{h}\frac{h}{\omega}\frac{\partial}{\partial t_i}\left(\frac{\omega}{\rho}-1\right)\,du_i^1du_i^2 \;=\; O(|t_i|(-\log|t_i|)^{-3})\int_{N^{i,+}_{s,t}}\frac{1}{|u|^4}\frac{1}{h}\,du_i^1du_i^2$$
$$= O(|t_i|). \tag{62}$$

Combining (59), (60), (61) and (62) along with the corresponding estimates for $N^{i,-}_{s,t}$, we obtain

$$\int_{\mathcal{N}^i_{s,t}}\frac{\partial}{\partial t_i}\left(\frac{\Phi_i\bar{\Phi}_i}{g^{hyp}_{s,t}}-\frac{\Phi_i\bar{\Phi}_i}{g^{gr}_{s,t}}\right)=O(|t_i|(-\log|t_i|)). \tag{63}$$

Next, we estimate the integral over $\mathcal{N}^l$, $l\neq i$. By (34), (57), Lemma 18, Lemma 31, Lemma 33 and the fact that outside of $\mathcal{N}^i_{s,t}$, $\frac{\partial\omega}{\partial t_i}=0$ and $\frac{\partial}{\partial t_i}\left(\frac{\omega}{\rho}\right)=W_i\left(\frac{\omega}{\rho}\right)$, we have

$$\int_{N^{l,+}_{s,t}}\frac{\frac{\partial\phi_i}{\partial t_i}\bar{\phi}_i}{h}\frac{h}{\omega}\left(\frac{\omega}{\rho}-1\right)\,du_l^1du_l^2 \;=\; O(|t_i|(-\log|t_l|)^{-2})\int_{N^{l,+}_{s,t}}\frac{1}{|u_l|^2}\frac{1}{h}du_l^1du_l^2$$
$$= O(|t_i|(-\log|t_l|)^{-2}), \tag{64}$$

$$\int_{N^{l,+}_{s,t}}\frac{\phi_i\bar{\phi}_i}{\omega}\left(-\frac{\frac{\partial\omega}{\partial t_i}}{\omega}\right)\left(\frac{\omega}{\rho}-1\right)\,du_l^1du_l^2=0, \tag{65}$$

$$\int_{N^{l,+}_{s,t}} \frac{\phi_i\bar{\phi}_i}{h}\frac{h}{\omega}\frac{\partial}{\partial t_i}\left(\frac{\omega}{\rho}-1\right)\,du_l^1du_l^2 \;=\; O(|t_i|(-\log|t_l|)^{-2})\int_{N^{l,+}_{s,t}}\frac{1}{|u_l|^2}\frac{1}{h}\,du_l^1du_l^2$$
$$= O(|t_i|(-\log|t_l|)^{-2}). \tag{66}$$

Combining (64), (65) and (66) along with the analogous estimates for $N^{l,-}_{s,t}$, we obtain that

$$\int_{\mathcal{N}^l_{s,t}}\frac{\partial}{\partial t_i}\left(\frac{\Phi_i\bar{\Phi}_i}{g^{hyp}_{s,t}}-\frac{\Phi_i\bar{\Phi}_i}{g^{gr}_{s,t}}\right)=O(|t_i|(-\log|t_l|)^{-2}). \tag{67}$$

The integral over $R_{s,t}\backslash\bigcup_l\mathcal{N}^l_{s,t}$ can be computed using the estimates of ϕ_i outside of $\bigcup_l\mathcal{N}^l_{s,t}$ contained in (57) and a similar argument. We obtain

$$\int_{R_{s,t}\backslash\bigcup_l\mathcal{N}^l_{s,t}}\frac{\partial}{\partial t_i}\left(\frac{\Phi_i\bar{\Phi}_i}{g^{hyp}_{s,t}}-\frac{\Phi_i\bar{\Phi}_i}{g^{gr}_{s,t}}\right)=O(|t_i|). \tag{68}$$

We finally compute the contribution from the boundary integrals in Lemma 34. Indeed, with $u_i = u_i^1 + iu_i^2$ and by using Lemma 18 and (34), we have

$$\begin{aligned}
&\int_{B_{s,t}^{i,+}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}} \left(\frac{\phi_i\bar{\phi}_i}{\rho} - \frac{\phi_i\bar{\phi}_i}{\omega}\right) du_i^1 du_i^2 \\
&= \int_{B_{s,t}^{i,+}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}} \frac{\phi_i\bar{\phi}_i}{h}\frac{h}{\omega}\left(\frac{\omega}{\rho} - 1\right) du_i^1 du_i^2 \\
&= O((-\log|t_i|)^{-2}) \int_{B_{s,t}^{i,+}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}} \left(\frac{\phi_i\bar{\phi}_i}{h} du_i^1 du_i^2\right) \qquad (69)\\
&= O((-\log|t_i|)^{-2}) \frac{i}{2} \int_{B_{s,t}^{i,+}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}} \left(\frac{\phi_i\bar{\phi}_i}{h} du_i d\bar{u}_i\right) \\
&= O((-\log|t_i|)^{-2}) \frac{i}{2} \int_{B_{s,t}^{i,+}} \eta(u_i)H(u_i)\frac{\phi_i\bar{\phi}_i}{h} d\bar{u}_i.
\end{aligned}$$

Moreover, with $\theta_i = \text{Arg } u_i$,

$$u_i = e^{i\theta_i}|t_i|^{\frac{1}{2}} \quad \text{and} \quad d\bar{u}_i = -ie^{-i\theta_i}d\theta_i|t_i|^{\frac{1}{2}} \;\Rightarrow\; u_i d\bar{u}_i = -i|t_i|d\theta_i$$

which together with (69), Lemma 28, (7) and (57) implies

$$\int_{B_{s,t}^{i,+}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}} \left(\frac{\phi_i\bar{\phi}_i}{\rho} - \frac{\phi_i\bar{\phi}_i}{\omega}\right) du_i^1 du_i^2 = O(|t_i|) \qquad (70)$$

and similarly for $B_{s,t}^{i,-}$. The first estimate follows from combining (63), (67), (68), (70) and Lemma 34.

Next, we prove the second estimate. We first compute the integral over $N_{s,t}^{i,+}$. By (34), (58), Lemma 18, Lemma 32 and Lemma 33,

$$\begin{aligned}
\int_{N_{s,t}^{i,+}} \frac{\frac{\partial\phi_I}{\partial t_i}\bar{\phi}_J}{h}\frac{h}{\omega}\left(\frac{\omega}{\rho} - 1\right) \; du_i^1 du_i^2 &= O((-\log|t_i|)^{-2}) \int_{N_{s,t}^{i,+}} \frac{1}{|u_i|^4}\frac{1}{h} du_i^1 du_i^2 \\
&= O(-\log|t_i|), \qquad (71)
\end{aligned}$$

$$\begin{aligned}
&\int_{N_{s,t}^{i,+}} \frac{\phi_I\bar{\phi}_J}{h}\frac{h}{\omega}\left(-\frac{\frac{\partial\omega}{\partial t_i}}{\omega}\right)\left(\frac{\omega}{\rho} - 1\right) \; du_i^1 du_i^2 \\
&= O(|t_i|^{-1}(-\log|t_i|)^{-3}) \int_{N_{s,t}^{i,+}} \frac{1}{|u_i|^2}\frac{1}{h} \; du_i^1 du_i^2 \\
&= O(|t_i|^{-1}(-\log|t_i|)^{-3}), \qquad (72)
\end{aligned}$$

$$
\begin{aligned}
&\int_{N^{i,+}_{s,t}} \frac{\phi_I\bar{\phi_J}}{h}\frac{h}{\omega}\frac{\partial}{\partial t_i}\left(\frac{\omega}{\rho}-1\right)\ du_i^1du_i^2 \\
&\quad = \quad O(|t_i|^{-1}(-\log|t_i|)^{-3})\int_{N^{i,+}_{s,t}}\frac{1}{|u_i|^2}\frac{1}{h}\ du_i^1du_i^2 \\
&\quad = \quad O(|t_i|^{-1}(-\log|t_i|)^{-3}). \qquad (73)
\end{aligned}
$$

Combining (71), (72) and (73) along with the corresponding estimate for $N^{i,-}_{s,t}$, we obtain

$$
\int_{\mathcal{N}^i_{s,t}}\frac{\partial}{\partial t_i}\left(\frac{\phi_I\bar{\phi_J}}{\rho}-\frac{\phi_I\bar{\phi_J}}{\omega}\right)du_i^1du_i^2 = O(|t_i|^{-1}(-\log|t_i|)^{-3}). \qquad (74)
$$

Next, we estimate the integral over $\mathcal{N}^l$, $l\neq i$. By (34), (58), Lemma 18, Lemma 31, Lemma 33 and the fact that outside of $\mathcal{N}^i_{s,t}$, $\frac{\partial\omega}{\partial t_i}=0$ and $\frac{\partial}{\partial t_i}\left(\frac{\omega}{\rho}\right)=W_i\left(\frac{\omega}{\rho}\right)$, we have

$$
\begin{aligned}
\int_{N^{l,+}_{s,t}}\frac{\frac{\partial\phi_I}{\partial t_i}\bar{\phi}_J}{h}\frac{h}{\omega}\left(\frac{\omega}{\rho}-1\right)\quad du_l^1du_l^2 &= O((-\log|t_l|)^{-2})\int_{N^{l,+}_{s,t}}\frac{1}{|u_l|}\frac{1}{h}du_l^1du_l^2 \\
&= O((-\log|t_l|)^{-2}), \qquad (75)
\end{aligned}
$$

$$
\int_{N^{l,+}_{s,t}}\frac{\phi_I\bar{\phi_J}}{\omega}\left(-\frac{\frac{\partial\omega}{\partial t_i}}{\omega}\right)\left(\frac{\omega}{\rho}-1\right)\ du_l^1du_l^2 = 0, \qquad (76)
$$

$$
\begin{aligned}
\int_{N^{l,+}_{s,t}}\frac{\phi_I\bar{\phi_J}}{h}\frac{h}{\omega}\frac{\partial}{\partial t_i}\left(\frac{\omega}{\rho}-1\right)\quad du_l^1du_l^2 &= O(|t_i|^{-1}(-\log|t_i|)^{-2})\int_{N^{l,+}_{s,t}}\frac{1}{h}\ du_l^1du_l^2 \\
&= O(|t_i|^{-1}(-\log|t_i|)^{-2}). \qquad (77)
\end{aligned}
$$

Combining (75), (76) and (77) along with the analogous estimates for $N^{l,-}_{s,t}$, we obtain that

$$
\int_{\mathcal{N}^l_{s,t}}\frac{\partial}{\partial t_i}\left(\frac{\phi_I\bar{\phi_J}}{\rho}-\frac{\phi_I\bar{\phi_J}}{\omega}\right)\ du_i^1du_i^2 = O(|t_i|^{-1}(-\log|t_i|)^{-3}). \qquad (78)
$$

The integral over $R_{s,t}\backslash\bigcup_l\mathcal{N}^l_{s,t}$ can be computed using the estimates of ϕ_I and ϕ_J outside of $\bigcup_l\mathcal{N}^l_{s,t}$ contained in (58) and a similar argument. We obtain

$$\int_{R_{s,t}\setminus\bigcup_l \mathcal{N}^l_{s,t}} \frac{\partial}{\partial t_i}\left(\frac{\phi_I\bar{\phi}_J}{\rho}-\frac{\phi_I\bar{\phi}_J}{\omega}\right) du_i^1 du_i^2 = O(|t_i|^{-1}(-\log|t_i|)^{-3}). \qquad (79)$$

We finally compute the contribution from the boundary integrals in Lemma 34. Indeed, as in (69) we have

$$\begin{aligned}
&\int_{B^{i,+}_{s,t}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\left(\frac{\phi_I\bar{\phi}_J}{\rho}-\frac{\phi_I\bar{\phi}_J}{\omega}\right) du_i^1 du_i^2 \\
&= \; O((-\log|t_i|)^{-2})\frac{i}{2}\int_{B^{i,+}_{s,t}} \eta(u_i)H(u_i)\frac{\phi_I\bar{\phi}_J}{h} d\bar{u}_i.
\end{aligned}$$

Moreover, by combining with Lemma 28, (7) and (58) this implies

$$\int_{B^{i,+}_{s,t}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\left(\frac{\phi_i\bar{\phi}_i}{\rho}-\frac{\phi_i\bar{\phi}_i}{\omega}\right) du_i^1 du_i^2 = O(1) \qquad (80)$$

and similarly for $B^{i,-}_{s,t}$. The second assertion follows from combining (74), (78), (79), (80) and Lemma 34. Q.E.D.

PROOF OF THEOREM 2. Below, we will provide the proof for estimates (i) and (v). The other estimates are proven by analogous arguments, and hence we omit their proofs.

First, we prove (i). By Lemma 35 it suffices to prove

$$\frac{\partial}{\partial t_i}\int_{R_{s,t}} \frac{\Phi_i\bar{\Phi}_i}{g^{gr}_{s,t}} = \frac{\partial}{\partial t_i}h^{ii} + O(|t_i|(-\log|t_i|)). \qquad (81)$$

By differentiating the identity (55) with respect to t_i, we have in $N^{i,+}_{s,t}$

$$\frac{\partial\phi_i}{\partial t_i} = \frac{1}{\pi u_i^2} + O(\frac{1}{|u_i|}).$$

Since $\phi_i = \frac{t_i}{\pi u_i^2} + O(\frac{|t_i|}{|u_i|})$

$$\phi_i\bar{\phi}_i = \frac{|t_i|^2}{\pi^2|u_i|^4} + O(\frac{|t_i|^2}{|u_i|^3}) \qquad (82)$$

and

$$\frac{\partial\phi_i}{\partial t_i}\bar{\phi}_i = \frac{\bar{t}_i}{\pi^2|u_i|^4} + O(\frac{|t_i|}{|u_i|^3}). \tag{83}$$

By (83), Lemma 18 and Corollary 33,

$$\begin{aligned}
\int_{N_{s,t}^{i,+}} \frac{\frac{\partial\phi_i}{\partial t_i}\bar{\phi}_i}{\omega} du_i^1 du_i^2 &= (1+O((-\log|t|)^{-2}))\int_{N_{s,t}^{i,+}} \frac{\frac{\partial\phi_i}{\partial t_i}\bar{\phi}_i}{h} du_i^1 du_i^2 \\
&= (1+O((-\log|t|)^{-2}))\left(\frac{\bar{t}_i}{\pi^2}\int_{N_{s,t}^{i,+}} \frac{1}{|u_i|^4}\frac{1}{h} du_i^1 du_i^2 \right.\\
&\quad \left. + O(|t_i|)\int_{N_{s,t}^{i,+}} \frac{1}{|u_i|^3}\frac{1}{h} du_i^1 du_i^2\right) \\
&= \frac{\bar{t}_i(-\log|t_i|)^3}{2\pi^3} + O(|t_i|(-\log|t_i|)). \qquad (84)
\end{aligned}$$

By (82) and Corrollary 33,

$$\begin{aligned}
\int_{N_{s,t}^{i,+}} \frac{\phi_i\bar{\phi}_i}{h}\frac{\frac{\partial h}{\partial t_i}}{h} du_i^1 du_i^2 &= \frac{|t_i|^2}{\pi^2}\int_{N_{s,t}^{i,+}} \frac{1}{|u_i|^4}\frac{1}{h}\frac{\frac{\partial h}{\partial t_i}}{h} du_i^1 du_i^2 \\
&\quad + O(|t_i|^2)\int_{N_{s,t}^{i,+}} \frac{1}{|u_i|^3}\frac{1}{h}\frac{\frac{\partial h}{\partial t_i}}{h} du_i^1 du_i^2 \\
&= \frac{\bar{t}_i(-\log|t_i|)^2}{4\pi^3} + O(|t_i|(-\log|t_i|)). \qquad (85)
\end{aligned}$$

By (85), (57) and Lemma 33,

$$\begin{aligned}
\int_{N_{s,t}^{i,+}} \frac{\phi_i\bar{\phi}_i}{\omega}\frac{\frac{\partial \omega}{\partial t_i}}{\omega} du_i^1 du_i^2 &= \int_{N_{s,t}^{i,+}} \frac{\phi_i\bar{\phi}_i}{h}\frac{\frac{\partial h}{\partial t_i}}{h} du_i^1 du_i^2 + \int_{N_{s,t}^{i,+}} \frac{\phi_i\bar{\phi}_i}{h}\left(\frac{\frac{\partial \omega}{\partial t_i}}{\omega} - \frac{\frac{\partial h}{\partial t_i}}{h}\right) du_i^1 du_i^2 \\
&= \frac{\bar{t}_i(-\log|t_i|)^2}{4\pi^3} + O(|t_i|(-\log|t_i|)) \\
&\quad + O(|t_i|(-\log|t_i|)^{-3})\int_{N_{s,t}^{i,+}} \frac{1}{|u_i|^4}\frac{1}{h} du_i^1 du_i^2 \\
&= \frac{\bar{t}_i(-\log|t_i|)^2}{4\pi^3} + O(|t_i|(-\log|t_i|)). \qquad (86)
\end{aligned}$$

Similarly we have the same formulas over $N^{i,-}_{s,t}$. Thus, by adding the above we obtain

$$\begin{aligned}&\int_{N^{i,+}_{s,t}} \frac{\partial}{\partial t_i}\left(\frac{\phi_i\bar{\phi}_i}{\omega}\right)\, du_i^1 du_i^2 + \int_{N^{i,-}_{s,t}} \frac{\partial}{\partial t_i}\left(\frac{\phi_i\bar{\phi}_i}{\omega}\right)\, du_i^1 du_i^2 \\ &= \frac{3\bar{t}_i(-\log|t_i|)^3}{2\pi^3} + \frac{\bar{t}_i(-\log|t_i|)^2}{2\pi^3} + O(|t_i|(-\log|t_i|)). \qquad (87)\end{aligned}$$

Since ω does not depend on t_i outside of $\mathcal{N}^i$, (57), Lemma 18 and Lemma 33 imply

$$\begin{aligned}\int_{N^{l,+}_{s,t}} \frac{\partial}{\partial t_i}\left(\frac{\phi_i\bar{\phi}_i}{\omega}\right)\, du_l^1 du_l^2 &= \int_{N^{l,+}_{s,t}} \frac{\frac{\partial\phi_i}{\partial t_i}\bar{\phi}_i}{\omega} du_l^1 du_l^2 \\ &= \int_{N^{l,+}_{s,t}} \frac{\frac{\partial\phi_i}{\partial t_i}\bar{\phi}_i}{h} du_l^1 du_l^2 \\ &= O(|t_i|)\int_{N^{l,+}_{s,t}} \frac{1}{|u_l|^2}\frac{1}{h} du_l^1 du_l^2 \\ &= O(|t_i|) \qquad (88)\end{aligned}$$

and similarly for $N^{l,-}_{s,t}$. Using the estimates of ϕ_i outside of $\bigcup_l \mathcal{N}^l_{s,t}$ contained in (57), we similarly obtain

$$\int_{R_{s,t}\setminus\bigcup_l \mathcal{N}^l_{s,t}} \frac{\partial}{\partial t_i}\left(\frac{\phi_i\bar{\phi}_i}{\omega}\right)\, du_i^1 du_i^2 = O(|t_i|). \qquad (89)$$

We finally compute the contribution from the boundary integrals in Lemma 34. Indeed, with $u_i = u_i^1 + iu_i^2$ and as in (69) we have

$$\begin{aligned}&\int_{B^{i,+}_{s,t}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\left(\frac{\phi_i\bar{\phi}_i}{\omega} du_i^1 du_i^2\right) \\ &= (1+O((-\log|t_i|)^{-2}))\frac{i}{2}\int_{B^{i,+}_{s,t}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\left(\frac{\phi_i\bar{\phi}_i}{h} du_i d\bar{u}_i\right) \\ &= (1+O((-\log|t_i|)^{-2}))\frac{i}{2}\int_{B^{i,+}_{s,t}} \eta(u_i)H(u_i)\frac{\phi_i\bar{\phi}_i}{h} d\bar{u}_i. \qquad (90)\end{aligned}$$

By (82), Lemma 28 and (7) since on the circle $B^{i,+}_{s,t}$ $|u_i| = |t_i|^{\frac{1}{2}}$, we have the estimates

$$H(u_i) = \frac{u_i}{t_i} + O(|t_i|^{-2\delta}),\quad \phi_i\bar{\phi}_i = \frac{|t_i|^2}{\pi^2|u_i|^4} + O(|t_i|^{\frac{1}{2}}) \quad \text{and} \quad \frac{1}{h} = \frac{(-\log|t_i|)^2|t_i|}{\pi^2}$$

Moreover, with $\theta_i = \text{Arg } u_i$ and as in (70),

$$\begin{aligned}
&\int_{B_{s,t}^{i,+}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}} \left(\frac{\phi_i \bar{\phi}_i}{\omega} du_i^1 du_i^2 \right) \\
&= \frac{\bar{t}_i(-\log|t_i|)^2}{2\pi^4} \int_0^{2\pi} \eta(|t_i|^{\frac{1}{2}} e^{i\theta_i}) d\theta_i + O(|t_i|). \qquad (91)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{B_{s,t}^{i,-}} i_{\tilde{\eta}\tilde{H}_i(v_i)\frac{\partial}{\partial v_i}} \left(\frac{\phi_i \bar{\phi}_i}{\omega} du_i^1 du_i^2 \right) \\
&= \frac{\bar{t}_i(-\log|t_i|)^2}{2\pi^4} \int_0^{2\pi} \tilde{\eta}(|t_i|^{\frac{1}{2}} e^{i\theta_i}) d\theta_i + O(|t_i|). \qquad (92)
\end{aligned}$$

On the other hand, since $\eta \equiv \frac{1}{2}$ on $B_{s,t}^{i,+}$ and $\tilde{\eta} = \frac{1}{2} + O(|t_i|^{\frac{1}{2}})$ on $B_{s,t}^{i,-}$ (cf. (53) and (54)), we obtain

$$\begin{aligned}
&\int_{B_{s,t}^{i,+}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}} \left(\frac{\phi_i \bar{\phi}_i}{\omega} du_i^1 du_i^2 \right) + \int_{B_{s,t}^{i,-}} i_{\tilde{\eta}\tilde{H}_i(v_i)\frac{\partial}{\partial v_i}} \left(\frac{\phi_i \bar{\phi}_i}{\omega} du_i^1 du_i^2 \right) = \\
&\qquad \frac{\bar{t}_i(-\log|t_i|)^2}{\pi^3} + O(|t_i|). \qquad (93)
\end{aligned}$$

Combining (87), (88), (89), (93) we obtain the desired estimate (81).

Next, we prove (v). By Lemma 35, it suffices to prove

$$\frac{\partial}{\partial t_i} \int_{R_{s,t}} \frac{\Phi_I \bar{\Phi}_J}{g_{s,t}^{gr}} = O(|t_i|^{-1}(-\log|t_i|)^{-3}). \qquad (94)$$

We write

$$\begin{aligned}
&\int_{N_{s,t}^{i,+}} \frac{\partial}{\partial t_i} \left(\frac{\phi_I \bar{\phi}_J}{\omega} \right) du_i^1 du_i^2 \\
&= \int_{N_{s,t}^{i,+}} \frac{\frac{\partial \phi_I}{\partial t_i} \bar{\phi}_J}{h} \frac{h}{\omega} du_i^1 du_i^2 - \int_{N_{s,t}^{i,+}} \frac{\phi_I \bar{\phi}_J}{h} \frac{h}{\omega} \frac{\frac{\partial \omega}{\partial t_i}}{\omega} du_i^1 du_i^2. \qquad (95)
\end{aligned}$$

Observe that by Lemma 33,

$$\int_{N_{s,t}^{i,+}} \frac{(-\log|u_i|)^2}{|u_i|^2} \frac{1}{h} du^1 du^2 \leq \int_{N_{s,t}^{i,+}} \frac{1}{|u_i|^3} \frac{1}{h} du_i^1 du_i^2 = O(1) \qquad (96)$$

By (58), Lemma 18, Lemma 33,

$$\begin{aligned}\int_{N_{s,t}^{i,+}} \frac{\frac{\partial \phi_I}{\partial t_i}\bar{\phi}_J}{h}\frac{h}{\omega}du_i^1du_i^2 &= O(1)\int_{N_{s,t}^{i,+}} \frac{1}{|u_i|^4}\frac{1}{h}du_i^1du_i^2 \\ &= O((-\log|t_i|)^3). \qquad (97)\end{aligned}$$

By (58), Lemma 18, Lemma 33, (96) and the fact that $O((-\log|u_i|)^2) = O(|u_i|^{-1})$,

$$\begin{aligned}&\int_{N_{s,t}^{i,+}} \frac{\phi_I\bar{\phi}_J}{h}\frac{h}{\omega}\frac{\frac{\partial\omega}{\partial t_i}}{\omega}du_i^1du_i^2 \\ &= O(|t_i|^{-1}(-\log|t_i|)^{-3}\int_{N_{s,t}^{i,+}} \frac{(-\log|u_i|)^2}{|u_i|^2}\frac{1}{h}du_i^1du_i^2 \\ &= O(|t_i|^{-1}(-\log|t_i|)^{-3}). \qquad (98)\end{aligned}$$

Combining (95), (97) and (98) with the analogous estimate on $N_{s,t}^{i,-}$,

$$\int_{\mathcal{N}_{s,t}^i} \frac{\partial}{\partial t_i}\left(\frac{\Phi_I\bar{\Phi}_J}{g_{s,t}^{gr}}\right) = O(|t_i|^{-1}(-\log|t_i|^{-3})). \qquad (99)$$

Since ω does not depend on t_i outside of $\mathcal{N}^i$, (58) and Lemma 33 imply

$$\begin{aligned}\int_{N_{s,t}^{l,+}} \frac{\partial}{\partial t_i}\left(\frac{\phi_I\bar{\phi}_J}{\omega}\right)\ du_l^1du_l^2 &= \int_{N_{s,t}^{l,+}} \frac{\frac{\partial\phi_I}{\partial t_i}\bar{\phi}_J}{\omega}du_l^1du_l^2 \\ &= O(1)\int_{N_{s,t}^{l,+}} \frac{1}{|u_l|}\frac{1}{h}du_l^1du_l^2 \\ &= O(1) \qquad (100)\end{aligned}$$

and similarly for $N_{s,t}^{l,-}$. Similarly,

$$\int_{R_{s,t}\setminus\cup\mathcal{N}_{s,t}^l} \frac{\partial}{\partial t_i}\left(\frac{\phi_I\bar{\phi}_J}{\omega}\right)\ du_l^1du_l^2 = O(1). \qquad (101)$$

We finally compute the contribution from the boundary integrals in Lemma 34. Indeed, as in (90)

$$\begin{aligned}&\int_{B_{s,t}^{i,+}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}}\left(\frac{\phi_I\bar{\phi}_J}{\omega}du_i^1du_i^2\right) \\ &= (1+O((-\log|t_i|)^{-2}))\frac{i}{2}\int_{B_{s,t}^{i,+}} \eta(u_i)H(u_i)\frac{\phi_I\bar{\phi}_J}{h}d\bar{u}_i. \qquad (102)\end{aligned}$$

By (58), Lemma 28 and (7) we have as before

$$\int_{B_{s,t}^{i,+}} i_{\eta H_i(u_i)\frac{\partial}{\partial u_i}} \left(\frac{\phi_I \bar{\phi}_J}{\omega} du_i^1 du_i^2 \right) = O((-\log |t_i|)^2). \tag{103}$$

and similarly, for $B_{s,t}^{i,-}$. Combining (99), (100), (101) and (103) we obtain the desired estimate (94). The other estimates are obtained analogously. Q.E.D.

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