MIDTERM FOR 110 405
FALL 2003

Answer all six questions. The first two questions are True/False. Fully justify your answer for the last four questions.

Question 1. (10 points; True/False). The sequence

$$x_n = \sum_{i=1}^{n} \frac{1}{i}$$

is Cauchy.

False (notice that $x_{n^2} - x_n = \sum_{i=n+1}^{n^2} \frac{1}{i} \geq (n^2 - n - 1)\, \frac{1}{n^2}$ does not go to zero as $n \to \infty$).

Question 2. (10 points; True/False). The sequence

$$y_n = \sum_{i=1}^{n} 2^{-i}$$

is Cauchy.

True. (It is a monotone sequence - since $y_{n+1} - y_n = 2^{-n-1}$ - and is bounded from above by 1. Hence, it converges to the limsup.)

Question 3. (20 points). Let T be the set of all Taylor series with 0's or 1's as coefficients, i.e., an element of T is an infinite series like

$$a_0 + a_1\, x + a_2\, x^2 + \cdots + a_n\, x^n + \cdots$$

where each a_i is either 0 or 1. Is T countable?

No, it is uncountable. To see this, note that T has the same cardinality as the power set 2^N. The map we use sends the above element of T to the subset of N which contains j if $a_{j-1} = 1$; this is clearly a bijection. Since 2^N is uncountable, so is T.

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Question 4. (20 points). Suppose that x_n is a Cauchy sequence. Prove that there exists a number $N \in \mathbf{N}$ so that $|x_j| \leq N$ for every j.

Since x_n is Cauchy, there exists $M \in \mathbf{N}$ so that for all $m, n > M$ we have

$$|x_n - x_m| < 1\,.$$

In particular, if $n > M$, we have by the triangle inequality that

$$|x_n| \leq |x_{M+1}| + |x_n - x_{M+1}| < |x_{M+1}| + 1\,.$$

It follows that *every* $|x_j|$ is bounded by the maximum of

$$\{|x_1|, \ldots, |x_M|,\ |x_{M+1}| + 1\}\,.$$

(This is finite since it is the maximum of a finite set of finite numbers!)

Question 5. (20 points). Let A be a set. Show that if x is a limit point of A, then there exists a sequence x_n of distinct points in A which converge to x.

Suppose that x is a L.P. of A. By definition, the nbhd

$$(x-1, x+1)$$

contains infinitely many points of A and, in particular, a point $x_1 \neq x$ belonging to A. Likewise, the nbhd

$$(x - |x - x_1|, x + |x - x_1|)$$

contains a point $x_2 \neq x$ belonging to A. Notice that $x_2 \neq x_1$! Now just continue this to get the sequence $x_n \to x$.

Question 6. (20 points). Suppose that $S \subset \mathbf{R}$ is an uncountable set. Prove that S contains at least one of its limit points.

If S contains no limit points, then for every $x \in S$ there is a ball $B_{r_x}(x)$ so that

$$B_{r_x}(x) \cap S = \{x\}\,.$$

In particular, these balls are disjoint. Since the rationals are dense, each ball contains some rational number. This gives an injective map from S to $\mathbf{Q}$ which is only possible when S is countable.