SOLUTIONS FOR MIDTERM FOR 110 405
FALL 2004

Answer all five questions. The first two questions are short answer. Fully justify your answer for the last three questions.

Question 1. (10 points; True/False). Every bounded sequence has a convergent subsequence; no proof is required.

True. For example, the sup is a limit point.

Question 2. (10 points; Short answer). Give an example of a set that is both open and closed; no proof is required.

Either $\mathbf{R}$ or the empty set $\emptyset$.

Question 3. (25 points). Suppose that E is a non-empty compact set. Show that $\sup(E)$ is contained in E.

- First, note that $\sup(E)$ exists (and is finite) since E is bounded.
- Claim: For any bounded set, we always have that $\sup E$ is either in E or a limit point of E.

  Proof of claim: Suppose that $\sup(E) \notin E$. Since $\sup(E)$ is the <u>least</u> upper bound, for every j, we get a point $y_j \in E$ with

  $$\sup(E) - 1/j < y_j < \sup(E)\,.$$

  (Otherwise $\sup(E) - 1/j$ would be a lower upper bound.) The sequence y_j converges to $\sup(E)$, proving the claim.
- Finally, since E is closed, it contains all of its limit points.

Question 4. (25 points). Suppose that for each λ in a set Λ, we have a positive real number $a_\lambda > 0$. Suppose also that for any natural number n and any $\lambda_1, \dots, \lambda_n \in \Lambda$ we have

$$\sum_{i=1}^{n} a_{\lambda_i} < 1\,.$$

Prove that the set Λ is at most countable (i.e, is either countable or finite).

- For each natural number n, define a set Λ_n by

  $$\Lambda_n = \{\lambda \in \Lambda \,|\, a_\lambda > 1/n\}\,.$$

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- The condition on the sums implies that Λ_n has less than n elements – in particular, each Λ_n is finite.
- Notice that, by the axiom of Archimedes, we can write Λ as the countable union of finite sets:
$$\Lambda = \cup_{n=1}^{\infty} \Lambda_n \,.$$
This means that Λ is also at most countable.

Question 5. (30 points). Suppose that a sequence y_n is defined iteratively by $y_0 = 1$ and then
$$y_{n+1} = \frac{1}{2 + y_n} \,.$$

(a) Compute the next three terms y_1, y_2, and y_3.
(b) Prove that the sequence y_n converges.

The first part is easy: 1/3, 3/7, and 7/17.

To see the convergence, notice that
$$y_{n+1} - y_n = \frac{1 - y_n(y_n + 2)}{y_n + 2} \,.$$
Similarly, using that $y_{n-1} = -2 + 1/y_n$, we get
$$y_n - y_{n-1} = \frac{y_n(y_n + 2) - 1}{y_n} \,.$$

Since the y_n's are always positive (prove this by induction if you are really conscientious), it follows that each $y_i < 1/2$ for $i > 0$.

Combining things, we get that
$$\frac{|y_{n+1} - y_n|}{|y_n - y_{n-1}|} = \frac{|y_n|}{|y_n + 2|} < \frac{1}{4} \,.$$
Since this ratio is less than one, the sequence must converge. See the solution of exercise 3 from page 54 for the proof of this last part (this was on problem set 3).