MATH 405 - FALL 2006 - MIDTERM EXAM -SOLUTION

Determine whether statements 1 through 10 are true of false. If you believe a statement is true, you **do not** *need to justify it. If you believe a statement is false, provide a counterexample. Each problem is worth 2 points and no partial credit will be given.*

1. There are no Cauchy sequences of positive rationals that are equivalent to a Cauchy sequence of negative rationals.

False. $1, \frac{1}{2}, \frac{1}{3}, ...$ and $-1, -\frac{1}{2}, -\frac{1}{3}, ...$ are equivalent.

2. If $x_1, x_2, ...$ and $y_1, y_2, ...$ are Cauchy sequences of real numbers with $y_i \neq 0$ for all i, then $\frac{x_1}{y_1}, \frac{x_2}{y_2}, ...$ is a Cauchy sequence.

False. If $x_n = 1$ for all n and $y_n = \frac{1}{n}$, then $\frac{x_n}{y_n} = n$ which is not Cauchy.

3. All sequences have a convergent subsequence.

False. 1,2,3,...

4. If A and B are sets of real numbers so that $\sup A < \inf B$, then the union of all intervals of the form (a, b) where $a \in A$ and $b \in B$ is open.

True. Union of any number of open sets is open.

5. If A and B are dense sets in $\mathbf{R}$, then $A \cap B$ is dense.

False. Let $A = \mathbf{Q}$ and $B = \mathbf{Q}^c$.

6. If $f(x)$ and $g(x)$ are Hölder continuous functions of order α, then $f(g(x))$ is a Hölder continuous function of order α.

False. Let $f(x) = x^{\frac{1}{2}}$ and $g(x) = x^{\frac{1}{2}}$, then $f(g(x)) = x^{\frac{1}{4}}$.

7. If f is a continuous function with domain $\mathbf{R}$ and U is open, then $f(U)$ is open.

False. Let $f(x) = 1$. Then $f(U) = \{1\}$ for any open set U.

8. If f is a continuous function with domain $\mathbf{R}$ and I is an open interval, then $f^{-1}(I)$ is an open interval.

False. Let $f(x) = \sin x$. Then $f^{-1}((-1/2, 1/2))$ is not an open interval. (It is a union of open intervals.)

9. If A is compact, then $\sup A \in A$ and $\inf A \in A$.

True.

10. If A is bounded but not compact, then $\sup A \notin A$ and $\inf A \notin A$.

False. Let $A = [0, 1) \cup (2, 3]$.

Prove the statements in 11 through 14. Each problem is worth 10 points and partial credit will be given. (*=easy, **=intermediate, ***=challenging)

11. The product of a negative real number and a positive real number is negative.

Let x be negative and y be positive. Then (by Definition 2.2.3) $-x$ is positive and (by Theorem 2.2.2) $(-x)y$ is positive. By the associative property of real numbers, $-(xy) = (-1)(xy) = (-x)y$ and hence $-(xy)$ is positive. This implies xy is negative (again by Definition 2.2.3).

Alternate solution (the long way): Let x be negative and y be positive. Then $-x$ is positive and there exists N_1, m_1 and a Cauchy sequence of rationals $a_1, a_2, \ldots$ so that $a_j \geq \frac{1}{N_1}$ for all $j \geq m_1$ (by Definition 2.2.3). Similarly, there exists N_2, m_2 and a Cauchy sequence of rationals $y_1, y_2, \ldots$ so that $a_j \geq \frac{1}{N_2}$ for all $j \geq m_2$. By Definition 2.2.2, $(-x)y$ is represented by $a_1y_1, a_2y_2, \ldots$. If we let $m = \max\{m_1, m_2\}$, then $a_jy_j \geq \frac{1}{N_1N_2}$ whenever $j \geq m$. Thus, $(-x)y$ is a positive number. Since $xy = -(-x)y$ (by associativity), xy is negative.

12. If f is a bounded, monotone increasing, continuous function defined on the interval [a,b), then f is uniformly continuous.

First, we claim that f has a limit from the left at b. Let $N \in \mathbf{N}$ be sufficiently large so that $b - \frac{1}{N} \in (a, b)$ and consider the sequence $f(b - \frac{1}{N+1}), f(b - \frac{1}{N+2}), f(b - \frac{1}{N+3}), \ldots$. Since f is monotone increasing and bounded, this sequence is monotone increasing and bounded. Therefore, it has a limit, say y_0. Let $n \in \mathbf{N}$ be given. Then there exists $k_0 \in \mathbf{N}$ so that $0 \leq y_0 - f(b - \frac{1}{k}) \leq \frac{1}{n}$ for all $k \geq k_0$. Since f is monotone increasing, $0 \leq y_0 - f(x) \leq \frac{1}{n}$ whenever $x \in (b - \frac{1}{k}, b)$. Thus, $|y_0 - f(x)| \leq \frac{1}{n}$ whenever $-\frac{1}{k} < x - b < 0$. This proves our claim.

Next, we let $g(x)$ be defined by setting $g(x) = f(x)$ for $x \in [a, b)$ and $g(b) = y_0$. Since f is continuous on the interval $[a, b)$ and $\lim_{x \to b^-} g(x) = \lim_{x \to b^-} f(x) = y_0 = g(b)$, g is continuous on the closed interval $[a, b]$. By Theorem 4.2.5, g is uniformly continuous and hence f is also (since $f(x) = g(x)$ on $[a, b)$).

13. If $\mathcal{B}$ is an open covering of a set A, then $\mathcal{B}$ has a countable subcover.

See the first paragraph of the proof of Theorem 3.3.2 and note that that portion of the proof does not need that A is compact.

14. If $\mathcal{O} = \{O_1, O_2, ...\}$ is a countable collection of open sets so that O_i is dense in $\mathbf{R}$ (for all $i = 1, 2, ...$), then the intersection of all the sets in $\mathcal{O}$ is dense in $\mathbf{R}$.

Let O be the intersection of the sets in $\mathcal{O}$. Assume O is not dense. Thus, there exists an open interval J which contains no points of O. Let I be an open interval so that closure$(I) \subset J$. Since O_1 is open and dense, $O_1 \cap I$ is open and nonempty. Thus, there exists a point x_1 and n_1 sufficiently small so that $I_1 = (x_1 - \frac{1}{n_1}, x_1 + \frac{1}{n_1}) \subset O_1 \cap I$. Since O_2 is open and dense, $O_2 \cap I_1$ is open and nonempty, there exists a point x_1 and $n_2 > n_1$ so that $I_2 = (x_2 - \frac{1}{n_2}, x_2 + \frac{1}{n_2}) \subset O_2 \cap I_1$. Continue inductively to construct a sequence

$$I_1 = (x_1 - \frac{1}{n_1}, x_2 - \frac{1}{n_2}), I_2 = (x_2 - \frac{1}{n_2}, x_2 + \frac{1}{n_2}), I_3 = (x_3 - \frac{1}{n_3}, x_2 + \frac{1}{n_3}), ...$$

so that $I_i \subset O_i$ and $I \supset I_1 \supset I_2 \supset I_3 ...$ and $n_1 < n_2 < n_3 < ....$ Since $x_j, x_k \in I_i$ whenever $j, k \geq i$, this implies $|x_j - x_k| < \frac{2}{n_i}$ whenever $j, k \geq n_i$ which in turn implies that $x_1, x_2, ...$ is a Cauchy sequence and converges to a number $x_0 \in$ closure(I). Furthermore, $x_0 \in \cap I_i \subset J$ and since $I_i \subset O_i$, we have $x_0 \in \cap O_i = O$. This is contradicts that J contains no points of O.