Solutions to Exam 1

1. Let $f(x, y) = x^2 + y^2 + xy - x + y$.

(a) Show that $x = 1, y = -1$ is the only critical point of f.

Solution: To find the critical point, set $f_x = 0$ and $f_y = 0$ to obtain the equations $2x + y - 1 = 0, 2y + x + 1 = 0$ which has the solution $x = 1, y = -1$.

(b) Use the second derivative test to show that $x = 1, y = -1$ is a local minimum (and thus an absolute minimum it is the only critical point) of f.

Solution: Since $f_{xx} = 2$ and $D = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 2 - 1 = 3 > 0$, the second derivative test says that f is a local minimum.

(c) Let $z = L(x, y)$ be the equation of the tangent plane of $z = f(x, y)$ at the critical point. Without evaluating any integrals, explain why the following inequality holds:

$$\int\int_R f(x, y)dA \geq \int\int_R L(x, y)dA$$

where R is any rectangle $[a, b] \times [c, d]$.

Solution: The tangent plane at $x = 1, y = -1$ is parallel to the xy-plane (since $f_x = 0 = f_y$ at that point) and has the equation of the form $L(x, y) = k$ where k is a constant equal to $f(1, -1)$. Since the local minimum $x = 1, y = -1$ is the only critical point of f, it is an absolute minimum and hence $f(x, y) \geq k$. This immediately implies that

$$\int\int_R f(x, y)dA \geq \int\int_R k\ dA = \int\int_R L(x, y)dA$$

2. Let $g_1(x, y, z) = x^2 + y^3 + z^4 - 2$ and $g_2(x, y, z) = xy - y^4 + z$.

(a) Let $h(u, v) = (u^2 + 1, v^2)$ and $g(x, y, z) = (g_1(x, y, z), g_2(x, y, z))$. Use the chain rule to compute the derivative of $h \circ g$ at point $x = 0, y = 1, z = 1$.

Solution: Let $h_1(u, v) = u^2 + 1$ and $h_2(u, v) = v^2$. Then

$$Dh = \begin{bmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 2u & 0 \\ 0 & 2v \end{bmatrix}$$

Additionally,

$$Dg = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 3y^2 & 4z^3 \\ y & x - 4y^3 & 1 \end{bmatrix}$$

At $x = 0, y = z, z = 1$, we have $u = g_1(0, 1, 1) = 0$, $v = g_2(0, 1, 1) = 0$, and hence,

$$\begin{aligned} D(h \circ g)(0, 1, 1) &= Dh(-2, 0)Dg(0, 1, 1) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(b) Let the surface S_1 be given by $g_1(x, y, z) = 0$ and the surface S_2 by $g_2(x, y, z) = 0$. Let the curve C be the intersection of S_1 and S_2. Show that the point $x = 0, y = 1, z = 1$ is *not* a critical point of the function $f(x, y, z) = 3xy^2 + y^2 + z^4$ restricted to C.

Solution: If $x = 0, y = 1, z = 1$ is a critical point, then

$$\nabla f(0,1,1) = \lambda_1 \nabla g_1(0,1,1) + \lambda_2 \nabla g_2(0,1,1) \qquad (*)$$

for some scalars λ_1, λ_2. This leads the following system of equations:

$$\begin{aligned} 3 &= \lambda_2 \\ 2 &= 3\lambda_1 - 4\lambda_2 \\ 4 &= 4\lambda_1 + \lambda_2 \end{aligned}$$

The first two equations imply $\lambda_1 = 14/3$ and $\lambda_2 = 3$. But plugging this into the third equation gives $4 = 4(14/3) + 3 = 65/3$, a contradiction.

(c) Give an example of a function $f(x, y, z)$ which, when restricted to C, has a critical point at $x = 0, y = 1, z = 1$.

Solution: We need a function satisfying equation (*) above for some scalars λ_1 and λ_1. For example, if we set $\lambda_1 = \lambda_2 = 1$, (*) gives

$$\nabla f(0,1,1) = (0,3,4) + (1,-4,1) = (1,-1,5)$$

Thus $f(x, y, z) = x - y + 5z$ will do. (Note: There are many possible answers.)

3. Let $f(x, y, z) = yz + xz - 6$.

(a) At the point $x = 1, y = 1, z = 1$, find the unit vector that points in the direction for which f is increasing at the fastest rate.

Solution: The function f increases at the fastest rate in the direction of

$$\frac{\nabla f(1,1,1)}{|\nabla f(1,1,1)|} = \frac{(1,1,2)}{\sqrt{6}}.$$

(b) For $\mathbf{x} = (1,1,1)$, find a vector $\mathbf{v}$ for which $\frac{d}{dt} f(\mathbf{x} + t\mathbf{v}) = 17$ at $t = 0$. (There are many possible answers. Just find one.)

Solution: If we let $\mathbf{v} = (a, b, c)$, then

$$17 = \frac{d}{dt}(f(\mathbf{x} + t\mathbf{v})|_{t=0} = \nabla f(1,1,1) \cdot \mathbf{v} = (1,1,2) \cdot (a,b,c) = a + b + 2c.$$

Thus, one possible answer is $\mathbf{v} = (17, 0, 0)$.

(c) Suppose $\mathbf{c}(t)$ is a flow line of ∇f with $\mathbf{c}(0) = (1,1,1)$. Calculate the acceleration of the curve $\mathbf{c}(t)$ at $t = 0$.

Solution: Since $\nabla f = (z, z, x + y)$, if we set $\mathbf{c}(t) = (x(t), y(t), z(t))$, then $\mathbf{c}'(t) = \nabla f(\mathbf{c}(t))$ implies

$$\begin{aligned} x'(t) &= z(t) \\ y'(t) &= z(t) \\ z'(t) &= x(t) + y(t) \end{aligned}$$

Thus,

$$\begin{array}{rclcl} x''(t) & = & z'(t) & = & x(t)+y(t) \\ y''(t) & = & z'(t) & = & x(t)+y(t) \\ z''(t) & = & x'(t)+y'(t) & = & 2z(t) \end{array}$$

and

$$\begin{array}{rclcl} x''(0) & = & x(0)+y(0) & = & 2 \\ y''(t) & = & x(0)+y(0) & = & 2 \\ z''(t) & = & 2z(0) & = & 2 \end{array}$$

4. For each of the four questions below, state whether the assertion is *true* or *false*. If it is true, *justify* and if it is false, *give a counterexample.*

(a) If $\mathbf{a}$ and $\mathbf{b}$ are vectors, the $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{b}$.

Solution: This is true. The vector triple product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}$ where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is given by the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

which is equal to zero since the last two rows coincide. Two vectors whose dot product is 0 are perpendicular.

(b) If $\mathbf{F}$ is a vector field, then $\nabla \times \mathbf{F}$ is perpendicular to $\mathbf{F}$.

Solution: This is flse. If $\mathbf{F}(x, y, z) = x\mathbf{i} + z\mathbf{j} + z\mathbf{k}$, then $\nabla \times \mathbf{F} = -\mathbf{i}$. On the other hand, $(\nabla \times \mathbf{F}) \cdot \mathbf{F} = -x$.

(c) If $\mathbf{F}, \mathbf{G}$ and $\mathbf{H}$ are vector fields so that $\mathbf{F}$ is a gradient field and $\mathbf{G}$ is a curl of some vector field, then $(\text{div } \mathbf{G})\mathbf{H} = \text{curl } \mathbf{F}$.

Solution: This is true. Since $\mathbf{G}$ is a curl of some vector field div $\mathbf{G} = 0$. Since $\mathbf{F}$ is a gradient field, curl $\mathbf{F} = \vec{0}$. So both the left hand side and the right hand side is equal to the zero vector.

(d) Assume $\nabla f(x, y, z) \neq \vec{0}$ for all (x, y, z). If $\mathbf{c}(t)$ is a flow line of ∇f, then the function $f(\mathbf{c}(t))$ is an increasing function of t.

Solution: This is true. Using the chain rule and the fact that $\mathbf{c}'(t) = \nabla f(\mathbf{c}(t))$, we have

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \mathbf{c}'(t) \cdot \mathbf{c}'(t) = |\mathbf{c}'(t)|^2 > 0$$

and hence $f(\mathbf{c}(t))$ is an increasing function.