An Introduction to Dynamical Systems

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Preface

The following text comprises the content of a course I have designed and have been running at Johns Hopkins University since the Spring of 2007. It is a senior (read: 400-level) analysis course in the basic tools, techniques, theory and development of what is sometimes called the modern theory of dynamical systems. The modern theory, as best as I can define it, is a focus on the study and structure of dynamical systems as little more than the study of the properties of one-parameter groups of transformations on a topological space, and what these transformations say about the properties of either the space or the group that is acting. It is a pure mathematical endeavor in that we study the material simply for the structure inherent in the constructions, and not for any particular application or outside influence. It is understood that many of the topics comprising this theory have natural, beautiful and important applications, some of which actually dictate the need for the analysis. But the true motivation for the study is little more than the fact that it is beautiful, rich in nuance and relevance in many tangential areas of mathematics, and that it is there.

When I originally pitched this course to the faculty here at Hopkins, there was no course like it in our department. We have a well-developed engineering school, filled with exceptionally bright and ambitious students, which along with strong natural and social science programs provide a ready audience for a course on the pure mathematical study of the theory behind what makes a mathematical model and why do we study them. We have a sister department here at Homewood, the Applied Mathematics and Statistics Department, which also offers a course in dynamical systems. However, their course seemed to focus on the nature and study of particular models that arise often in other classes, and then to mine those models for relevant information to better understand them. But as a student of the field, I understood that a course on the very nature of using functions as models and then studying their properties in terms of the dynamical information inherent in them was currently missing from our collective curriculum. Hence the birth of this course.

In my personal and humble opinion, it continues to be difficult to find a good text that satisfies all of the properties I think would constitute the perfect text for a course such as this one: (1) a focus on the pure mathematical theory of the abstract dynamical system, (2) advanced enough that the course can utilize the relevant topological, analytical and algebraic nature of the topic without requiring so much prerequisite knowledge as to limit enrollment to just mathematicians, (3) rich enough to develop a good strong story to tell which provides a solid foundation for later individual study, and (4) basic enough so that students in the natural sciences and engineering can access the entirety of the content given only the basic foundational material of vector calculus, linear algebra and differential equations.

It is a tall order, this is understood. However, it can be accomplished, I believe, and this text is my attempt at accomplishment.

The original text I chose for the course is the text *A First Course in Dynamics*, by Boris Hasselblatt and Anatole Katok (Cambridge University Press: 2003). A wonderfully designed story-line from two transformational mathematicians in the field, I saw the development line they took, from the notion of simple dynamics to the more complicated, as proper and intuitive. I think their focus on using the properties of functions and that of the spaces they are acting upon to develop material is the correct one for this "modern" approach. And their reuse of particular examples over and over again as the story progresses is a strong one. However, in the years I have been teaching and revising the course, I have found myself, adding material, redesigning the focus and the schedule, and building in a slightly different storyline. All of this diverging from the text. Encouraged by my students and my general thrill at the field, I decided to create my version of a text. This manuscript is this version.

What the reader will find in this text is my view of the basic foundational ideas that comprise a first (and one semester) course in the modern theory of dynamical systems. It is geared toward the upper-level undergraduate student studying either mathematics, or engineering or the natural and social sciences with a strong emphasis in learning the theory the way a mathematician would want to teach the theory. It is a proof-based course. However, when I teach the course, I do understand that some of my students do not have experience in writing mathematics in general and proofs in particular. Hence I use the content of the course as a way to also introduce these students to the development of ideas instead of just calculation. It is my hope that these students, upon finishing this course, will begin to look at the models and analysis they see in their other applied classes with an eye to the nature of the model and not just to its mechanics. They are studying to be scholars in their chosen field. Their ability to really "see" the mathematical structure of their tools will be necessary for them to contribute to their field.

This course (this text) is designed to be accessible to a student who has had a good foundational course in the following:

- vector calculus, at least up to the topics of surface integration and the "big three" theorems of Green, Stokes and Gauss;
- linear algebra, through linear transformations, kernels and images, eigenspaces, orthonormal bases and symmetric matrices; and
- differential equations, with general first and second order equations, linear systems theory, nonlinear analysis, existence and uniqueness of first order solutions, and the like.

While I make it clear in my class that analysis and algebra are not necessary prerequisites, this course cannot run without a solid knowledge of the convergence of general sequences in a space, the properties of what makes a set a topological space, and the workings of a group. Hence in the text we introduce these ideas as needed, sometimes through development and sometimes simply through introduction and use. I have found that most of these advanced topics are readily used and workable for students even if they are not fully explored within the confines of a university course. Certainly, having sat through courses in advanced algebra and analysis will be beneficial, but I believe they are not necessary. The text to follow, like all proper endeavors in mathematics, should be seen as a work in progress. The storyline, similar to that of Hasselblatt and Katok, is to begin with basic definitions of just what is a dynamical system. Once the idea of the dynamical content of a function or differential equation is established, we take the reader a number of topics and examples, starting with the notion of simple dynamical systems to the more complicated, all the while, developing the language and tools to allow the study to continue. Where possible and illustrative, we bring in applications to base our mathematical study in a more general context, and to provide the reader with examples of the contributing influence the sciences has had on the general theory. We pepper the sections with exercises to broaden the scope of the topic in current discussion, and to extend the theory into areas thought to be of tangential interest to the reader. And we end the text at a place where the course I teach ends, on a notion of dynamical complexity, topological entropy, which is still a active area of research. It is my hope that this last topic can serve as a landing on which to begin a more individualized, higher-level study, allowing the reader to further their scholarly endeavor now that the basics have been established.

I am thankful to the mathematical community for facilitating this work, both here at Hopkins and beyond. And I hope that this text contributes to the learning of high-level mathematics by both students of mathematics as well as students whose study requires mathematical prowess.

CHAPTER 1

What is a Dynamical System?

1.1. Definitions

As a mathematical discipline, the study of dynamical systems most likely originated at the end of the 19th century through the work of Henri Poincare in his study of celestial mechanics (footnote this: See Scholarpedia[History of DS]). Once the equations describing the movement of the planets around the sun are formulated (that is, once the mathematical model is constructed), looking for solutions as a means to describe the planets' motion and make predictions of positions in time is the next step. But when finding solutions to sets of equations is seemingly too complicated or impossible, one is left with studying the mathematical structure of the model to somehow and creatively narrow down the possible solution functions. This view of studying the nature and structure of the equations in a mathematical model for clues as to the nature and structure of its solutions is the general idea behind the techniques and theory of what we now call dynamical systems. Being only a 100+ years old, the mathematical concept of a dynamical system is a relatively new idea. And since it really is a focused study of the nature of functions of a single (usually),real (usually) independent variable, it is a subdiscipline of what mathematicians call real analysis. However, one can say that dynamical systems draws its theory and techniques from many areas of mathematics, from analysis to geometry and topology, and into algebra. One might call mathematical areas like geometry, topology and dynamics second generation mathematics, since they tend to bridge other more pure areas in their theories. But as the study of what is actually means to model phenomena via functions and equations, dynamical systems is sometimes called the mathematical study of any mathematical concept that evolves over time. So as a means to define this concept more precisely, we begin with arguably a most general and yet least helpful statement:

Definition 1.1. A *dynamical system* is a mathematical formalization for any *fixed rule* which describes the dependence of the position of a point in some *ambient space* on a *parameter*.

- The *parameter* here is usually called "time", and can be
  (1) discrete (think the natural numbers $\mathbb{N}$ or the integers $\mathbb{Z}$), or
  (2) continuous (defined by some interval in $\mathbb{R}$).

  It can also be much more general, taking values as subsets of $\mathbb{C}$, $\mathbb{R}^n$, the quaternions, or indeed any set with the structure of a group. However, classically speaking, a dynamical system really involves a parameter that takes values only in a subset of $\mathbb{R}$. We will hold to this convention.
- The *ambient space* has a state to it in the sense that all of its points have a marked position which can change as one varies the parameter. Roughly, every point has a position relative to the other points and (generalized)

coordinates often provide this notion of position. Fixing the coordinates and allowing the parameter to vary, one can create a functional relationship between the points at one value of the parameter and those at another parameter value. In general, this notion of relative point positions in a space and functional relationships on that space involves the notion of a topology on a set. A topology gives a set the mathematical property of a space; It endows the elements of a set with a notion of nearness to each other and allows for functions on a set to have properties like continuity, differentiability, and such. We will expound more on this later. We call this ambient space the *state space*: it is the set of all possible states a dynamical system can be in at any parameter value (at any moment of time.)

- The *fixed rule* is usually a recipe for going from one state to the next in the ordering specified by the parameter. For discrete dynamical systems, it is often given as a function, defining the dynamical system recursively. In continuous systems, where it is more involved to define what the successor to a parameter value may be, the continuous movement of points in a space may be defined by a differential equation, the solution of which would be a function involving both the points and the parameter and taking values back in the state space. Often, the latter function is called the *evolution* of the system, providing a way of going from any particular state to any other state reachable from that initial state via a value of the parameter. As we will see, such a function can be shown to exist, and its properties can often be studied, but in general, it will NOT be known *a priori*, or even knowable *a posteriori*.

While this idea of a dynamical system is far too general to be very useful, it is instructive. Before creating a more constructive definition, let's look at some classical examples:

1.1.1. Ordinary Differential Equations (ODEs). Given the first-order (vector)-ODE in $\mathbb{R}^n$,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t),$$

a solution, if it exists, is a vector of functions $\mathbf{x}(t) = [x_1(t) \;\; x_2(t) \;\; \cdots \;\; x_n(t)]^T$ parameterized by a real variable $t \in \mathbb{R}$ where the common domain of t is some subinterval of $\mathbb{R}$. Here:

- The ODE itself is the fixed rule, describing the infinitesimal way to go from one state to the next by an infinitesimal change in the value of the parameter t. Solving the ODE means finding the unknown function $\mathbf{x}(t)$, at least up to a set of constants determined by some initial state of the system. The inclusion of initial data provide this initial state of the variables of the system, making the system an Initial Value Problem (IVP). A solution to an IVP, $\mathbf{x}(t)$, for valid values of t, provides the various "other" states that the system can reach (either forward or backward in time) as compared to the initial state. Collecting up all the functions $\mathbf{x}(t)$ for all valid sets of initial data (basically, finding the expression that writes the constants of integration of the general solution to the ODE in terms of the initial data vairables), into one big function IS the evolution.

- This type of a dynamical system is called *continuous*, since the parameter t will take values in some domain (an interval) in $\mathbb{R}$. Dynamical systems like this arising from ODEs are also called *flows*, since the various IVP solutions in phase space look like the flow lines of a fluid in phase space flowing along the slope field (vector field defined by the ODE).
- In this particular example, the state space is (a subset of) $\mathbb{R}^n$, and the solutions live as parameterized curves in the state space. The solution curves are called trajectories. We also call this state space the *phase space*.

REMARK 1.2. One should be careful about not confusing a state space, the space of all possible states of a system, with the *configuration space* of, say, a physical system governed by Newton's Second Law of Motion. For example, the set of all positions of a pendulum at any moment of time is simply the circle. This would be the configuration space, the space of all possible configurations. But without knowing the velocity of the pendulum at any particular configuration, one cannot predict future configurations of the system. The state space, in the case of the pendulum, involves both the position of the pendulum and its velocity (we will see why in a later chapter.) For a standard ODE system like the general one above, the state space, phase space and configuration space all coincide. We will elaborate more on this later.

Place a picture here of a pendulum at a configuration with a small velocity and one with the same configuration with a large velocity. One second later, the pendulums are in different positions. So ODEs are examples of continuous dynamical systems (actually differentiable dynamical systems). Solving the ODE (finding the vector of functions $\mathbf{x}(t)$), means finding the rule which stipulates any state of a point in some other parameter value given the state of the point at a starting state. But as we will soon see, when thinking of ODEs as dynamical systems, we have a different perspective on what we are looking for in solving the ODE.

1.1.2. Maps. Given any set X and a function $f : X \to X$ from X to itself, one can form a dynamical system by simply applying the function over and over (iteratively) to X. When the set has a topology on it (a mathematically precise notion of an "open subset", allowing us to talk about the positions of points in relation to each other), we can then discuss whether the function f is continuous or not. When X has a topology, it is called a space, and a continuous function $f : X \to X$ is called a *map*.

- We will always assume that the sets we specify in our examples are spaces, but will detail the topology only as needed. Mostly they will exist as subsets of real space $\mathbb{R}^n$, where the notion of nearness comes from a precise definition of a distance between points given by a metric. In this context, there should be little confusion. Here the *state space* is X, with the positions of its points given by coordinates on X (defined by the topology.
- the fixed rule is the map f, which is also sometimes called a *cascade*.
- In a purely formal way, f defines the evolution (recursively) by composing f with itself. Indeed, $x \in X$, define $x_0 = x$, and $x_1 = f(x_0)$. Then

$$x_2 = f(x_1) = f(f(x_0) = f^2(x_0),$$

and for all $n \in \mathbb{N}$, (the natural numbers)

$$x_n = f(x_{n-1}) = f(f(x_{n-2})) = \overbrace{f(f(\cdots f(f(x_0))\cdots))}^{n \text{ times}} = f^n(x_0).$$

- Maps are examples of *discrete dynamical systems*. Some examples of discrete dynamical systems you may have heard of include discretized ODEs, including difference equations and time-t maps, and fractal constructions like Julia sets and the associated Mandelbrot arising from maps of the complex plane to itself. Some objects that are not considered to be constructed by dynamical systems (at least not directly) include fractals like Sierpinski's carpet, Cantor sets, and Fibonacci's Rabbits (given by a second order recursion). Again, we will get to these.

Besides these classic ideas of a dynamical system, there are much more abstract notions of a dynamical system:

1.1.3. Symbolic Dynamics. Given a set of symbols $M = \{A, B, C, \ldots\}$, consider the "space" of all bi-infinite sequences of these symbols (infinite on both sides)

$$\Omega_M = \left\{ (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \Big| \, i \in \mathbb{Z}, x_i \in M \right\}.$$

One can consider Ω_M as the space of all functions from $\mathbb{Z}$ to M. Now let $f : \Omega_M \to \Omega_M$ be the *shift* map: on each sequence, it simply takes $i \mapsto i + 1$; each sequence goes to another sequence which looks like a shift of the original one.

NOTE. *We can always consider this (very large) set of infinite sequences as a space once we give it a topology like I mentioned. This would involve defining open subsets for this set, and we can do this through ϵ-balls by defining a notion of distance between sequences (a metric). For those who know analysis, what would be a good metric for this set to make it a space using the metric topology? For now, simply think of this example as something to think about. Later and in context, we will focus on this type of dynamical system and it will make more sense.*

This discrete dynamical system is sometimes used as a new dynamical system to study the properties of an old dynamical system whose properties were hard to study. We will revisit this later.

Sometimes, in a time-dependent system, the actual dynamical system will need to be constructed before it can be studied.

1.1.4. Billiards. Consider two point-beads moving at constant (possibly different) speeds along a finite length wire, with perfectly elastic collisions both with each other and with the walls. A state of this system will be the positions of each of the beads at a given moment of time.

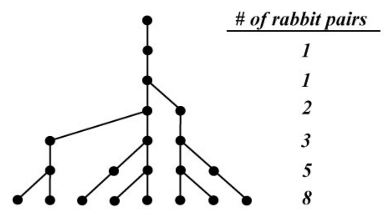
EXERCISE 1. One way to view the state space is as a triangle in the plane. Work this out. What are the vertices of this triangle? Does it accurately describe ALL of the states of the system? Are the edges of the triabgle part of the state space? And once you correctly describe the state space, what will motion look like in it? How does the dynamical system evolve?

Now consider a point-ball moving at a constant velocity inside a closed, bounded region of $\mathbb{R}^2$, where the boundary is smooth and collisions with the boundary are perfectly elastic. Questions:

(1) How does the shape of the region affect the types of paths the ball can traverse?
(2) Are there closed paths (periodic ones)?
(3) can there be a dense path (one that eventually gets arbitrarily close to any particular point in the region?

There is a method to study this type of dynamical system called a billiard by creating a discrete dynamical system to record movement and collecting only essential information. In this discrete dynamical system, regardless of the shape of the region, the state space is a cylinder. Can you see it? If so, what would be the evolution?

1.1.5. Other recursions. The Rabbits of Leonardo of Pisa is a beautiful example of a type of growth that is not exponential, but something called asymptotically exponential. We will explore this more later. For now, though, we give a brief description: Place a newborn pair of breeding rabbits in a closed environment. Rabbits of this species produce another pair of rabbits each month after they become fertile (and they never die nor do they experience menopause). Each new pair of rabbits (again, neglect the incest, gender and DNA issues) becomes fertile after a month and starts producing each month starting in the second month. How many rabbits are there after 10 years?



Month	a_n	j_n	b_n	total pairs
1	0	0	1	1
2	0	1	0	1
3	1	0	1	2
4	1	1	1	3
5	2	1	2	5
6	3	2	3	8
7	5	3	5	13

Given the chart in months, we see a way to fashion an expression governing the number of pairs at the end of any given month: Start with r_n, the number of pairs of rabbits in the nth month. Rabbits here will come in three types: Adults a_n, juveniles j_n, and newborns b_n, so that $r_n = a_n + j_n + b_n$. Looking at the chart, we can see that there are constraints on these numbers:

(1) the number of newborns at the $(n+1)$st stage equals the number of adults at the nth stage plus the number of juveniles at the nth stage, so that
$$b_{n+1} = a_n + j_n.$$
(2) This is also precisely equal to the number of adults at the $(n+1)$st stage, so that
$$a_{n+1} = a_n + j_n.$$
(3) and finally, the number of juveniles a the $(n+1)$st stage is just the number of newborns at the nth stage, so that
$$j_{n+1} = b_n.$$

Thus, we have
$$r_n = a_n + j_n + b_n = (a_{n-1} + j_{n-1}) + b_{n-1} + (a_{n-1} + j_{n-1}).$$

And since in the last set of parentheses, we have $a_{n-1} = a_{n-2} + j_{n-2}$ and $j_{n-1} = b_{n-2}$, we can substitute these in to get

$$r_n = a_n + j_n + b_n = (a_{n-1} + j_{n-1}) + b_{n-1} + (a_{n-1} + j_{n-1})$$
$$= a_{n-1} + j_{n-1} + b_{n-1} + a_{n-2} + j_{n-2} + b_{n-2} = r_{n-1} + r_{n-2}.$$

Hence the pattern is ruled by a second-order recursion $r_n = r_{n-1} + r_{n-2}$ with initial data $r_0 = r_1 = 1$. Being a second order recursion, we cannot go to the next state from a current state without also knowing the previous state. This is an example of a model which is not a dynamical system. We can make it one, but we will need a bit more structure, which we will introduce later.

Now, with this general idea of what a dynamical system actually is, along with numerous examples, we give a much more accurate and useful definition of a dynamical system:

Definition 1.3. A dynamical system is a triple $(\mathcal{S}, \mathcal{T}, \Phi)$, where $\mathcal{S}$ is the state space (or phase space), $\mathcal{T}$ is the parameter space, and

$$\Phi : (\mathcal{S} \times \mathcal{T}) \longrightarrow \mathcal{S}$$

is the evolution.

Some notes:

- In the previous discussion, the fixed rule was a map or an ODE which would only define recursively what the evolution would be. In this definition, Φ defines the entire system, mapping where each point $s \in \mathcal{S}$ goes for each parameter value $\tau \in \mathcal{T}$. It is the functional form of the fixed rule, unraveling the recursion and allowing one to go from a starting point to any point reachable by that point given a value of the parameter.
- In ODEs, Φ plays the role of the *general* solution, as a 1-parameter family of solutions (literally a 1-parameter family of transformations of phase space): In this general solution, one knows for ANY specified starting value where it will be for ANY valid parameter value, all in one function of two variables.

Example 1.4. In the Malthusian growth model, $\dot{x} = kx$, with $k \in \mathbb{R}$, and $x(t) \geq 0$ a population, the general solution is given by $x(t) = x_0 e^{kt}$, for $x_0 \in \mathbb{R}_0^+ = [0, \infty)$, the unspecified initial value at $t = 0$. (The notation $\mathbb{R}_0^+$ comes from the strictly positive real numbers $\mathbb{R}^+$ together with the value 0.) Really, the model works for $x_0 \in \mathbb{R}$, but if the model represents population growth, then initial populations can ONLY be nonnegative, right? Here, $\mathcal{S} = \mathbb{R}_0^+$, $\mathcal{T} = \mathbb{R}$ and $\Phi(s, t) = se^{kt}$.

Example 1.5. Let $\dot{x} = -x^2 t$, $x(0) = x_0 > 0$. Using the technique commonly referred to as separation of variables, we can integrate to find an expression for the general solution as $x(t) = \frac{1}{\frac{t^2}{2} + C}$. And since $x_0 = \frac{1}{C}$ (you should definitely do these calculations explicitly!), we get

$$\Phi(x_0, t) = \frac{1}{\frac{t^2}{2} + \frac{1}{x_0}} = \frac{2x_0}{x_0 t^2 + 2}.$$

Here, we are given $\mathcal{S} = \mathbb{R}^+$, and we can choose $\mathcal{T} = \mathbb{R}$. Question: Do you see any issues with allowing $x_0 < 0$? Let $x_0 = -2$, and describe the particular solution on the interval $t \in (0, 2)$.

Exercise 2. Integrate to find the general solution above for the Initial Value Problem $\dot{x} = -x^2 t$, $x(0) = x_0 > 0$.

- In discrete dynamics, for a map $f : X \to X$, we would need a single expression to write $\Phi(x, n) = f^n(x)$. This is not always easy or doable, as it would involve finding a functional form for a recursive relation. Try doing this with f a general polynomial of degree more than 1.

Example 1.6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = rx$, for $r \in \mathbb{R}_+$. Then $\Phi(x, n) = r^n x$.

Example 1.7. For Leonardo of Pisa's (also known as Fibonacci, in case you recognized the pattern of the sequence) rabbits, we will have to use the recursion to calculate every month's population to get to the 10-year mark. However, if we could find a *functional* form for the recursion, giving population in terms of month, we could than simply plug in $12 \cdot 10 = 120$ months to calculate the population after 10 years. The latter functional form is the evolution Φ in the definition of a dynamical system above. How does one find this? We will see.

Exercise 3. Find a closed form expression for the evolution of $f(x) = rx + a$, in the case where $-1 < r < 1$ and a are constants. Also determine the unique point where $f(x) = x$ in this case.

Exercise 4. For $g(x) = x^2+1$, write out the first four iterates $g^i(x)$, $i = 1, 2, 3, 4$. Then look for a pattern with which to write out the nth iterate, $g^n(x)$. Do you see the difficulty? Now overcome it and write an expression for the evolution.

In general, finding Φ (in essence, solving the dynamical system) is very difficult if not impossible, and certainly often impractical and/or tedious. However, it is often the case that the purpose of studying a dynamical system is not to actually solve it. Rather, it is to gain insight as to the structure of its solutions. Really, we are trying to make *qualitative* statements about the system rather than quantitative ones. Think about what you did when studying nonlinear systems of first order ODEs in any standard undergraduate course in differential equations. Think about what you did when studying autonomous first order ODEs.

Before embarking on a more systematic exploration of dynamical systems, here is another less rigorous definition of a dynamical system:

Definition 1.8. Dynamical Systems as a field of study attempts to understand the structure of a changing mathematical system by identifying and analyzing the things that do not change.

There are many ways to identify and classify this notion of an unchanging quantity amidst a changing system. But the general idea is that if a quantity within a system does not change while the system as a whole is evolving, then that quantity holds a special status as a symmetry. Identifying symmetries can allow one to create a new system, simpler than the previous, where the symmetry has been factored out, either reducing the number of variables or the size of the system.

More specifically, here are some of the more common notions:

- Invariance: First integrals: Sometimes a quantity, defined as a function on all or part of the phase space, is constant along the solution curves of the system. If one could create a new coordinate system of phase space where one coordinate is the value of the first integral, then the solution curves become simply the constant values of this coordinate. The coordinate becomes useless to the system, and it can be discarded. Thus the new system has less degrees of freedom than the original system. Phase space volume: In a conservative vector field, as we will see, if we take a small ball of points of a certain volume and then flow along the solution curves to the vector field, the ball of points will typically bend and stretch in very complicated ways. But it will remain an open set, and its total volume will remain the same. This is phase volume preservation, and it says a lot about the behavior and types of solution curves.

- Symmetry: Periodicity: Sometimes solution curves are closed, and solutions retrace their steps over certain intervals of time. If the entire system behaves like this, the direction of the flow contains limited information about the solution curves of the system. One can in a sense factor out the periodicity, revealing more about the remaining directions of the state space. Or even near a singular periodic solution, one can discretize the system at the period of the periodic orbit. This discretized system has a lower order, or number of variables, then the original.
- Asymptotics: In certain systems where the time is not explicitly expressed in the system, one can start at any moment in time and the evolution depends only on the starting time. In systems like these, the long-term behavior of solutions may be more important than where they are in any particular moment in time. IN a sense, one studies the asymptotics of the system, instead of attempting to solve. Special solutions like equilibria and limit cycles are easy to find, and their properties become important elements of the analysis.

EXAMPLE 1.9. In an exact differential equation

$$M(x,y)\,dx + N(x,y)\,dy = M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

we have $M_y = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = N_x$. We know then that there exists a function $\phi(x,y)$, where $\frac{\partial \phi}{\partial x} = M$ and $\frac{\partial \phi}{\partial y} = N$. Indeed, given a twice differentiable function $\phi(x,y)$ defined on a domain in the plane, it's level sets are equations $\phi(x,y) = C$, for C a real constant. Each level set defines y implicitly as a function of x. Thinking of y as tied to x implicitly, differentiate $\phi(x,y) = C$ with respect to x and get

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\,\frac{dy}{dx} = 0.$$

This last equation will match the original ODE precisely if the two above properties hold. The interpretation then is: The solutions to the ODE correspond to the level sets of the function ϕ. We can say that that solutions to the ODE "are forced to live" on the level sets of ϕ. Thus, we can write the general solution set (at least implicitly) as $\phi(x,y) = C$, again a 1-parameter family of solutions. Here ϕ is a *first integral* of the flow given by the ODE.

EXERCISE 5. Solve the differential equation $12 - 3x^2 + (4-2y)\frac{dy}{dx} = 0$ and express the general solution in terms of the initial condition $y(x_0) = y_0$. This is your function $\phi(x,y)$.

EXAMPLE 1.10. Newton-Raphson: Finding a root of a (twice differentiable) function $f : \mathbb{R} \to \mathbb{R}$ leads to a discrete dynamical system $x_n = g(x_{n-1})$, where

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

One here does not need to actually solve the dynamical system (find a form for the function Φ). Instead, all that is needed is to satisfy some basic properties of f to know that if you start sufficiently close to a root, the long-term (asymptotic) behavior of any starting point IS a root.

EXERCISE 6. One can use the Intermediate Value Theorem in single variable calculus to conclude that there is a root to the polynomial $f(x) = x^3 - 3x + 1$ in the unit interval $I = [0, 1]$ (check this!). For starting values every tenth on I, iterate $g(x)$ to estimate this root to three decimal places (it converges quite quickly!). Now try to explain what is happening when you get to both $x_0 = .9$ and $x_0 = 1$.

EXAMPLE 1.11. Autonomous ODEs: One can integrate the autonomous first-order ODE

$$y' = f(y) = (y-2)(y+1), \quad y(0) = y_0,$$

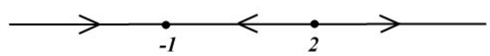
since it is separable, and the integration will involve a bit of partial fraction decomposing. The solution is

$$y(t) = \frac{Ce^{3t} + 2}{1 - Ce^{3t}}. \tag{1.1.1}$$

EXERCISE 7. Calculate Equation 1.1.1 for the ODE in Example 1.11.

EXERCISE 8. Now find the evolution for the ODE in Example 1.11 (this means write the general solution in terms of y_0 instead of the constant of integration C.)

But really, is the explicit solution necessary? One can simply draw the phase line,



From this schematic view of the long-term tendencies of each solution, one can glean a lot of information about the solutions of the equation. For instance, the equilibrium solutions occur at $y(t) \equiv -1$ and $y(t) \equiv 2$, and that the equilibrium at -1 is asymptotically stable (the one at 2 is unstable). Thus, if long-term behavior is all that is necessary to understand the system, then we have:

$$\lim_{t\to\infty} y(t) = \begin{cases} -1 & \text{if} \quad y_0 < 2 \\ 2 & \text{if} \quad y_0 = 2 \\ \infty & \text{if} \quad y_0 > 2. \end{cases}$$

In both these last two examples, actually solving the dynamical system isn't necessary to gain important and possibly sufficient information about the system.

1.2. The viewpoint

Dynamical Systems, as a field of study, is a type of mathematical analysis; the study of the formal properties of the real number system and the structures defined on them (think sets, functions, spaces, etc.) Indeed, the properties of functions and the spaces that serve as their domains (and codomains) are intimately intertwined in sometimes obvious and often subtle ways. For example, in a celebrated theorem by Luitzen E. J. Brouwer, any continuous function from a compact, convex space to itself must contain at least one point where its image under the function is the same as the point itself. This property has enormous implications for not simply the function we apply to the space, but for the space itself. The consequences of a theorem like this are evident even on the beginning stages of math, like calculus and differential equations.

In general, studying how a map moves around the points of the space is to study the *dynamical content* of the map. Where the points go, upon repeated iteration of a map on a space, or how solutions of a differential equation behave once their parameter domain is known is to study the system *dynamically.* If most or all of the solutions tend to look alike, or if the diversity of the ways a collection of iterates of a point under a map is small, then we say that the dynamics are *simple.* In essence, they are easy to describe, or it does not take a lot of information to describe them. In contrast, if different solutions to the ODE can do many different things, or if it takes a lot of information to describe how many different ways a map can move distinct points around in a space, we say that the dynamics are *complex* or *complicated.* One may say that a dynamical system is more interesting if it is more complicated to describe, although that is certainly a subjective term.

Solving a dynamical system, or finding an explicit expression for the evolution, is typically not the general goal of an analysis of a dynamical system. Many non-linear systems of ODEs are difficult if not impossible to solve. Rather, the goal of an analysis of a dynamical system is the general description of the movement of points under the map or the ODE.

In the following chapters, we will develop a language and methods of analysis to study the dynamical content of various kinds of dynamical systems. We will survey both discrete and continuous dynamical systems that exhibit a host of phenomena, and mine these situations for ways to classify and characterize the behavior of the iterates of a map (or solutions of the ODE). We will show how the properties of the maps and the spaces they use as domains affect the dynamics of their interaction. We will start with situations that display relatively simple dynamics, and progress through situations and applications of increasing complexity (complicated behavior). In all of these situations, we will keep the maps and spaces as easy to define and work with as possible, to keep the focus directly on the dynamics.

Perhaps the best way to end this chapter is on a more philosophical note, and allow a possible raison d'etre for why dynamical systems even exists as a field of study enmeshed in the world of analysis, topology and geometry:

DEFINITION 1.12. Dynamical systems is the study of the information contained in and the effects of groups of transformations of a space.

For a discrete dynamical system defined by a map on a space, the properties of the map as well as those of the space, will affect how points are moved around the space. As we will see, maps with certain properties can only do certain things, and if the space has a particular property, like the compact, convex space above, then certain things must be true (or may not), like a fixed-point free transformation. Dynamics is the exploration of these ideas, and we will take this view throughout this text.

CHAPTER 2

Simple Dynamics

2.1. Preliminaries

2.1.1. A simple system. To motivate our first discussion and set the playing field for an exploration of some simple dynamical systems, recall some general theory of first-order autonomous ODEs in one dimension: Let

$$\dot{x} = f(x), \quad x(0) = x_0$$

be an IVP (again, an ODE with an initial value) where the function $f(x)$ is a differentiable function on all of $\mathbb{R}$. From any standard course in differential equations, this means that solutions will exist and be uniquely defined for all values of $t \in \mathbb{R}$ near $t = 0$ and for all values of $x_0 \in \mathbb{R}$. Recall that the general solution of this ODE will be a 1-parameter family of functions $x(t)$ parameterized by x_0. In reality, one would first use some sort of integration technique (as best as one can; remember this ODE is always separable, although $\frac{1}{f(x)}$ may not be easy to integrate. As an example, consider $f(x) = e^{x^2}$) to find $x(t)$ parameterized by some constant of integration C. Then one would solve for the value of C given a value of x_0. Indeed, one could solve generally for C as a function of x_0, and then substitute this into the general solution, to get

$$x(t, x_0) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

as the evolution. Then, for each choice of x_0, we would get a function $x_{x_0}(t) : \mathbb{R} \to \mathbb{R}$ as the particular solution to the IVP. We will use the notation with a subscript for x_0 to accentuate that the role of x_0 is that of a parameter. Specifying a value means solving the IVP for that value of x_0. Leaving x_0 unspecified means that we are looking for a particular solution at a fixed value of x_0. The resulting graph of $x_{x_0}(t)$ would "live" in the tx-plane as a curve (the trajectory) passing through the point $(0, x_0)$. Graphing a bunch of representative trajectories gives a good idea of what the evolution looks like. You did this in your differential equations course when you created phase portraits.

Example 2.1. Let $\dot{x} = kx$, with $k \in \mathbb{R}$ a constant. Here, a general solution to the ODE is given by $x(t) = Ce^{kt}$. If, instead, we were given the IVP $\dot{x} = kx$, $x(0) = x_0$, the particular solution would be $x(t) = x_0 e^{kt}$. The trajectories would look like graphs of standard exponential functions (as long as $k \neq 0$) in the tx-plane. Below in Figure 1 are the three cases which look substantially different from each other: When $k > 0$, $k = 0$, and $k > 0$.

Recall in higher dimensions, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, we typically do not graph solutions explicitly as functions of t. Rather, we use the t-parameterization of solutions $\mathbf{x}(t) = \left[x_1(t) \ x_2(t) \ \cdots \ x_n(t)\right]^T$ to trace out a curve directly in the $\mathbf{x}$-space. This space, whose coordinates are the set of dependent variables $x_1, x_2, \ldots, x_n$, is called

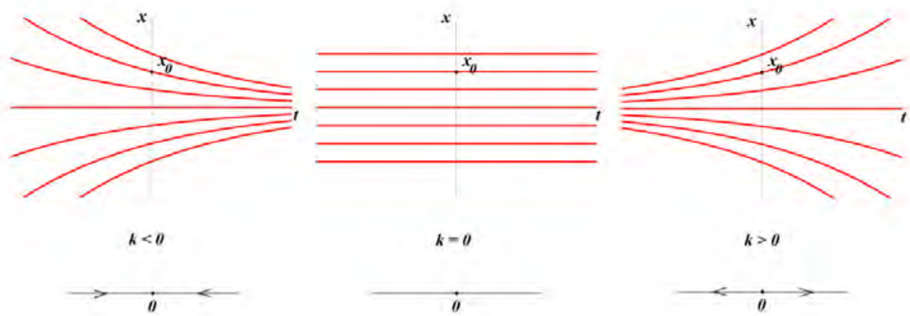


FIGURE 1. Sample solutions for $x_{x_0}(t) = x_0 e^{kt}$.

the phase-space (sometimes the tx-plane from above, or more generally the $t\mathbf{x}$-space is called the *trajectory space* to mark the distinction). The diagrams in the plane that correspond to linear systems with a saddle at the origin, or a spiral sink are examples of phase planes with representative trajectories. Often, particularly in phase space, trajectories are also called *orbits*.

EXAMPLE 2.2. The linear system IVP $\dot{x} = -y$, $\dot{y} = x$, $x(0) = 1$, $y(0) = 0$ has the particular solution $x(t) = \cos t$, $y(t) = \sin t$. Graphing the trajectory, according to the above, means graphing the curve in the txy-space, a copy of $\mathbb{R}^3$. While informative, it may be a little tricky to fully "see" what is going on. But the orbit, graphed in the xy-plane, which is the phase space, is the familiar unit circle (circle of radius 1 centered at the origin). Here $t \in \mathbb{R}$ is the coordinate directly on the circle, and even the fact that it overwrites itself infinitely often is not a serious sacrifice to understanding. See Figure 2.

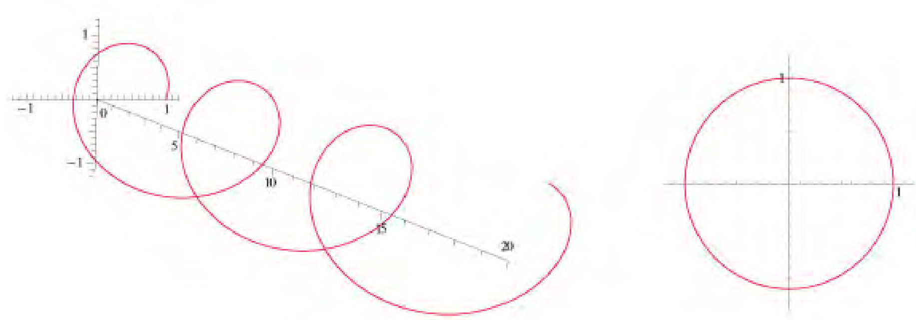


FIGURE 2. Solution curve $x(t) = \cos t$, $y(t) = \sin(t)$ in trajectory space and the phase plane.

Even for autonomous ODEs in one-dependent variable, we designed a schematic diagram called a phase-line to give a qualitative description of the "motion" of solutions to $\dot{x} = f(x)$.

EXAMPLE 2.3. The phase lines for $\dot{x} = kx$ for the three cases in Figure 1 are below the graphs. The proper way to think of these lines is as simply a copy of the vertical axis (the x-axis in this case of the tx-plane) in each of the graphs, marking the equilibrium solutions as special points, and indicating the direction of change of the x-variable as t increases. All relevant information about the long-term behavior is encoded in these phase lines. In fact, these lines ARE the 1-dimensional phase spaces of the ODE, and the arrows simply indicate the direction of the parameterized $x(t)$ inside the line. It is hard to actually see the parameterized curves, since they all run over the top of each other. This is why we graph solutions in 1-variable ODEs using t explicitly, while for ODEs in two or more dependent variables, we graph using t implicitly, as the coordinate directly ON the curve in the phase space.

2.1.2. The time-t map. Again, for $\dot{x} = f(x)$, $x(0) = x_0$, the general solution $x(t, x_0) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a 1-parameter family of solutions, written as $x_{x_0}(t)$, parameterized by x_0. However, we can also think of this family of curves in a much more powerful way: As a 1-parameter family of transformations of the phase space! To see this, rewrite the general solution as $\varphi(t, x_0) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ instead of the possibly confusing notation $x(t, x_0)$. Now instead of thinking of x_0 as the parameter, fixing the second argument and varying the first as the independent variable, do it the other way: Fix a value of t, and allow the variable $x_0 = x$ (the starting point) to vary. Then we get for $t = t_0$:

$$\varphi(t_0, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \varphi_{t_0}(x) : \mathbb{R} \to \mathbb{R}, \quad x(0) \longrightarrow x(t_0).$$

As t varies, every point $x \in \mathbb{R}$ (thought of as the initial point $x(0)$, gets "mapped" to its new position at $x(t_0)$. Since all solutions are uniquely defined, this is a function for each value of t_0, and will have some very nice properties. But this alternate way of looking at the solutions of an ODE, as a family of transformations of its phase space, is the true *dynamical view*, and one we will explore frequently.

Place a picture of how points move around phase space at time t_0. Viewing this as solely a transformation of phase space is the dynamical view.

Let X denote any particular topological space. For now, though, just think of X as some subset of the real space $\mathbb{R}^n$, something you are familiar with.

DEFINITION 2.4. For $f : X \to X$ a map, define the set

$$\mathcal{O}_x = \left\{ y \in X \;\middle|\; y = f^n(x), \;\; n \in \mathbb{N} \right\}$$

as the (forward) orbit of $x \in X$ under f.

Some notes:

- If f is invertible, we can also then define the backward orbit for $n \in -\mathbb{N}$, or the full orbit for $n \in \mathbb{Z}$.
- We can also write $\mathcal{O}_x = \{x, f(x), f^2(x), \ldots\}$, or for $x_{n+1} = f(x_n)$, $\mathcal{O}_x = \{x_0, x_1, x_2, \ldots\}$.
- When it makes sense for clarity, we may use the notation $\mathcal{O}_x^+$ for the forward orbit, $\mathcal{O}_x^-$ for the backward orbit, and then $\mathcal{O}_x$ for the full orbit. This will usually be understood in context, though.

Consider the discrete dynamical system $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = rx$, $r > 0$. What do the orbits look like? Basically, for $x \in \mathbb{R}$, we get

$$\mathcal{O}_x = \left\{ x, rx, r^2x, r^3x, \dots, r^nx, \dots \right\}.$$

In fact, we can "solve" this dynamical system by constructing the evolution

$$\Phi(x, n) = r^n x.$$

Do the orbits change in nature as one varies the value of r? How about when r is allowed to be negative? How does this relate to the ordinary differential equation $\dot{x} = kx$?

Definition 2.5. For $t \geq 0$, the *time-t* map of a continuous dynamical system is the transformation of state space which takes $x(0)$ to $x(t)$.

Example 2.6. Let $k < 0$ in $\dot{x} = kx$, with $x(0) = x_0$. Here, the state space is $\mathbb{R}$ (the phase space, as opposed to the trajectory space $\mathbb{R}^2$), and the general solution is $\Phi(x_0, t) = x_0 e^{kt}$ (the evolution of the dynamical system is $\Phi(x, t) = xe^{kt}$. Notice that

$$\Phi(x, 0) = x, \quad \text{while} \quad \Phi(x, 1) = e^k x.$$

Hence the time-1 map is simply multiplication by $r = e^k$. The time-1 map is the discrete dynamical system on $\mathbb{R}$ given by the function above $f(x) = rx$. In this case, $r = e^k$, where $k < 0$, so that $0 < r = e^k < 1$. See Figure 2.1.2. Now how do the orbits behave?

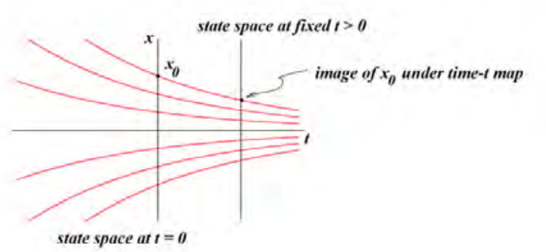


Figure 3. The time-t map for some positive time of $\dot{x} = kx$, $k < 0$.

Exercise 9. Given any dynamical system, describe the time-0 map.

Definition 2.7. For a discrete dynamical system $f : X \to X$, a *fixed point* is a point $x_* \in X$, where $f(x_*) = x_*$, or where

$$\mathcal{O}_{x_*} = \left\{ x_*, x_*, x_*, \dots \right\}.$$

The orbit of a fixed point is also called a trivial orbit. All other orbits are called non-trivial.

In our example above, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^k x$, $k < 0$, we have $x = 0$ as the ONLY fixed point. This corresponds nicely with the unique particular solution to the ODE $\dot{x} = kx$ corresponding to the equilibrium $x(t) \equiv 0$.

So what else can we say about the "structure" of the orbits? That is, what else can we say about the "dynamics" of this dynamical system? For starters, the forward orbit of a given x_0 will look like the graph of the discrete function $f_{x_0} : \mathbb{N} \to \mathbb{R}^2, f_{x_0}(n) = x_0 e^{kn}$. Notice how this orbit follows the trajectory of x_0 of the continuous dynamical system $\dot{x} = kx$. Here, f is the time-1 map of the ODE.

Notice also that, as a transformation of phase space (the x-axis), f is not just a continuous function but a differentiable one, with $0 < f'(x) = e^k < 1$, $\forall x \in \mathbb{R}$. The orbit of the fixed point at $x = 0$, as a sequence, certainly converges to 0. But here ALL orbits have this property, and we can say

$$\forall x \in \mathbb{R}, \quad \lim_{n\to\infty} \mathcal{O}_x = 0, \text{ or } \mathcal{O}_x \longrightarrow 0.$$

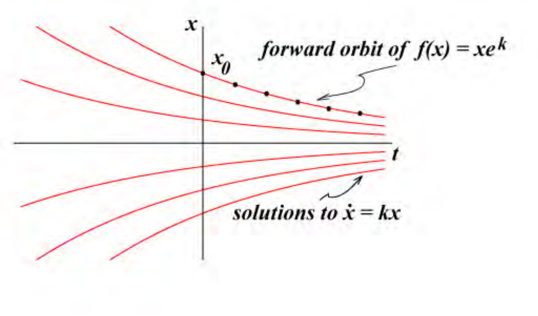


FIGURE 4. The forward orbit of $f(x) = xe^k$ lives on a solution to $\dot{x} = kx$, $k < 0$.

This gives a sense of what we will mean by a dynamical system exhibiting *simple dynamics*: If with very little effort or additional structure, one can completely describe the nature of all of the orbits of the system. Here, there is one fixed point, and all orbits converge to this fixed point.

DEFINITION 2.8. For a discrete dynamical system, a smooth curve (or set of curves) ℓ in state space is called an *orbit line* if $\forall x \in \ell$, $\mathcal{O}_x \subset \ell$.

EXAMPLE 2.9. The orbit lines for time-t maps of ODEs are the trajectories of the ODE.

EXERCISE 10. Go back to Figure 1. Describe completely the orbit structure of the discrete dynamical system $f(x) = rx$ for other two cases, when $r = 1$ and $r > 1$ (corresponding to $r = e^k$, for $k = 0$ and $k > 0$, respectively). That is, classify all possible different types of orbits, in terms of whether they are fixed or not, where they go as sequences, and such. You will find that even here, the dynamics are simple, but at least for the $k > 0$ case, one has to be a little more careful about accurately describing where orbits go.

EXERCISE 11. As in the previous exercise, describe the dynamics of the discrete dynamical system $f(x) = rx$, when $r < 0$ (again, there are cases here). In particular, what do the orbit lines look like in this case? You will find that this case does not, in general, correspond to a time-t map of the ODE $\dot{x} = kx$ for any value of k (why not?)

EXERCISE 12. Show that there does not exist a first-order, autonomous ODE where the map $f(x) = rx$ corresponds to the time-1 map, when $r < 0$.

EXERCISE 13. Construct a second-order ODE whose time-1 map is $f(x) = rx$, where $r < 0$ is any given constant.

EXERCISE 14. For the discrete dynamical system $f : \mathbb{R} \to \mathbb{R}$, $f(x) = rx + b$, calculate the evolution in closed form. Then completely describe the orbit structure when $b \neq 0$, noting in particular the different cases for different values of $r \in \mathbb{R}$.

EXERCISE 15. Given $\dot{x} = f(x)$, $f \in C^1(\mathbb{R})$, recall that an *equilibrium solution* is defined as a constant function $x(t) \equiv c$ which solves the ODE. They can be found by solving $f(x) = 0$ (remember this?) Instead, define an equilibrium solution $x(t)$ as follows: A solution $x(t)$ to $\dot{x} = f(x)$ is called an *equilibrium solution* if there exists $t_1 \neq t_2$ in the domain of $x(t)$ where $x(t_1) = x(t_2)$. Show that this new definition is equivalent to the old one.

EXERCISE 16. For the first-order autonomous ODE $\frac{dp}{dt} = \frac{p}{2} - 450$, do the following:

- Solve the ODE by separating variables. Justify explicitly why the absolute value signs are not necessary when writing the general solution as a single expression.
- Calculate the time-1 map for this ODE flow.
- Discuss the simple dynamics of this discrete dynamical system given by the time-1 map.

2.1.3. Contractions. The above questions are all good to explore. For now, the above example $f(x) = e^k x$, where $k < 0$, is an excellent example of a particular class of dynamical systems which we will discuss presently.

DEFINITION 2.10. A *metric* on a subset of Euclidean space $X \subset \mathbb{R}^n$ is a function $d : X \times X \to \mathbb{R}$ where

(1) $d(x, y) \geq 0$, $\forall x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$, $\forall x, y \in X$.
(3) $d(x, y) + d(y, z) \geq d(x, z)$, $\forall x, y, z \in X$.

One such choice of metric is the "standard Euclidean distance" metric

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Note that for $n = 1$, this metric reduces to $d(x, y) = \sqrt{(x - y)^2} = |x - y|$.

EXERCISE 17. Explicitly show that the standard Euclidean distance metric is indeed a metric by showing that it satisfies the three conditions.

EXERCISE 18. On $\mathbb{R}^n$, define a notion of distance by $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|$. Show this notion of distance is a metric. (For $n = 2$, this is sometimes called the taxicab or the Manhattan metric. Can you see why?)

EXERCISE 19. On $\mathbb{R}^2$, consider a notion of distance defined by the following:

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} |x_1 - y_1| & \text{if} \quad x_1 \neq y_1 \\ |x_2 - y_2| & \text{if} \quad x_1 = y_1. \end{cases}$$

This is similar to a lexicographical ordering of points in the plane. Show that this notion of distance is NOT a metric on $\mathbb{R}^2$.

EXERCISE 20. The original definition of a circle as a planar figure comes directly from Euclid himself: A *circle* is the set of points in the plane equidistant from a particular point. Naturally, using the Euclidean metric, a circle is what you know well as a circle. Show that circles in the taxicab metric on $\mathbb{R}^2$ are squares whose diagonals are parallel to the coordinate axes.

EXERCISE 21. Following on the previous exercise, construct a metric on $\mathbb{R}^2$ whose circles are squares whose sides are parallel to the coordinate axes. (Hint: Rotate the taxicab metric.)

EXERCISE 22. Let S be a circle of radius $r > 0$ centered at the origin of $\mathbb{R}^2$. It's circumference is $2\pi r$. Euclidean distance in the plane does restrict to a metric directly on the circle. Here instead, construct a metric on the circle using arc-length, and verify that it is a metric. (Be careful about measuring distance correctly.)

REMARK 2.11. When discussing points in Euclidean space, it is conventional to denote scalars (elements of $\mathbb{R}$) with a variable in italics, and vectors (elements of $\mathbb{R}^n$, $n > 1$) as a variable in boldface. Thus $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. In the above definition of a metric, we didn't specify whether X was a subset of $\mathbb{R}$ or something larger. In the absence of more information regarding a space X, we will always use simple italics for its points, so that $x \in X$, even if it is possible that $X = \mathbb{R}^5$, for example. We will only resort to the vector notation when it is assured that we are specifically talking about vectors of a certain size. This is common in higher mathematics like topology.

DEFINITION 2.12. A map $f : X \to X$, where $X \subset \mathbb{R}^n$ is called *Lipschitz continuous* (with constant λ), or λ-Lipschitz, if

$$d\left(f(x), f(y)\right) \leq \lambda d(x, y), \quad \forall x, y \in X. \tag{2.1.1}$$

Some notes:

- The set X can always inherit the metric on $\mathbb{R}^n$ simply by declaring that the distance between two points in X is defined by the their distance in $\mathbb{R}^n$ (See Exercise 22). So subsets of $\mathbb{R}^n$ are always metric spaces. One can always define a different metric on X if one wants (the gist of the exercise above). But the fact that X is a metric space comes for free, as they sometimes say.
- λ is a bound on the stretching ability (comparing the distances between the images of points in relation to the distance between their original positions) of f on X. This is actually a form of smoothness stronger than continuity: Lipschitz functions are always continuous, but there are continuous functions that are not Lipschitz.

  EXERCISE 23. Show for $f : \mathbb{R} \to \mathbb{R}$ that Lipschitz continuity implies continuity.

  EXERCISE 24. Let $f(x) = \frac{1}{x}$. Show f is Lipschitz continuous on any domain (a, b), $a > 0$, $a < b \leq \infty$, and for any particular choice of a and b, produce the constant λ. Then show that f is not Lipschitz continuous on $(0, \infty)$.

  EXERCISE 25. For a given non-negative λ, construct a function whose domain in all of $\mathbb{R}$, that is precisely λ-Lipschitz continuous on $I = (-\infty, 2) \cup (2, \infty)$ but not Lipschitz continuous.

  EXERCISE 26. Produce a function that is continuous on $I = [-1, 1]$ but not Lipschitz continuous there.

- To get a sense for what Lipschitz continuity is saying, consider the following: On a bounded interval in $\mathbb{R}$, polynomials are always Lipschitz continuous. Rational functions, on the other hand, even though they are continuous and differentiable on their domains, are not Lipschitz continuous on any interval whose closure contains a vertical asymptote.

- It should be obvious that $\lambda > 0$. Why?
- We can define

$$\operatorname{Lip}(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)},$$

which is the infimum of all λ's that satisfy Equation 2.1.1. When we speak of specific values of λ for a λ-Lipschitz function, we typically use $\lambda = \operatorname{Lip}(f)$, if known.

PROPOSITION 2.13. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on an open interval I, where $\forall x \in I$, we have $|f'(x)| \leq \lambda$. Then f is λ-Lipschitz.*

PROOF. Really, this is simply an application of the Mean Value Theorem: For a function f differentiable on a bounded, open interval (a,b) and continuous on its closure, there is at least one point $c \in (a,b)$ where $f'(c) = \frac{f(b)-f(a)}{b-a}$, the average total change of the function over $[a,b]$. Here then, for any $x,y \in I$ (thus ALL of $[x,y] \in I$ even when I is neither closed nor bounded), there will be at least one $c \in I$ where

$$d(f(x), f(y)) = |f(x) - f(y)| = |f'(c)||x-y| \leq \lambda |x-y| = \lambda d(x,y).$$

□

DEFINITION 2.14. A λ-Lipschitz function $f : X \to X$ on a metric space X is called a *contraction* if $\lambda < 1$.

EXAMPLE 2.15. Back to the previous example $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^k x$, the time-1 map of the ODE $\dot{x} = kx$. Given that $f'(x) = e^k$ everywhere, in the case that $k < 0$, the map f is a contraction on ALL of $\mathbb{R}$. Indeed, using the Euclidean metric in $\mathbb{R}$, we have

$$d(f(x), f(y) = \left|e^k x - e^k y\right| = \left|e^k(x-y)\right| = \left|e^k\right||x-y| = e^k|x-y| = \lambda|x-y|$$

for all $x, y \in \mathbb{R}$, where $\lambda = e^k < 1$.

EXERCISE 27. Without using derivative information, show that $f(x) = ax + b$ is a-Lipschitz on $\mathbb{R}$.

EXERCISE 28. Again without using derivative information, show that the monomial x^n, $n \in \mathbb{N}$ is na^{n-1}-Lipschitz on the interval $[0,a]$

EXERCISE 29. Find a for the largest interval $[0,a]$ where $f(x) = 3x^2 - 2$ is a contraction.

Before we continue, we need to clarify some of the properties of the intervals we will be using in our dynamical systems. Here are a couple of definitions:

DEFINITION 2.16. A subset $U \in \mathbb{R}$ is called bounded if there exists a number $M > 0$ so that $\forall x \in U$, we have $|x| < M$.

DEFINITION 2.17. An interval I is called *closed* in $\mathbb{R}$ if it contains all of its limit points. If the interval is bounded (as a subset of $\mathbb{R}$), then this means that I includes its endpoints. But closed intervals need not be bounded. Hence closed intervals in $\mathbb{R}$ take one of the forms $[a,b]$, $(-\infty,b]$, $[a,\infty)$ or $(-\infty,\infty)$, for $-\infty < a \leq b < \infty$.

PROPOSITION 2.18. *Let $f : I \to I$ for I a closed, bounded interval, and f continuously differentiable (we understand f to be one-sided differentiable at the endpoints) with $|f'(x)| < 1$ $\forall x \in I$. Then f is a contraction.*