HOMEWORK PROBLEM SET 7: DUE MARCH 27, 2019

AS.110.211 HONORS MULTIVARIABLE CALCULUS
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Question 1. Show that $D^2_{\mathbf{v}}(\mathbf{a}) = \mathbf{v}^T Hf(\mathbf{a})\mathbf{v}$. That is, show that the second directional derivative of a C^2, real-valued function $f : \mathbb{R}^n \to \mathbb{R}$, evaluated at $\mathbf{a} \in X$ and in the direction of $\mathbf{v}$, is the quadratic form given by the Hessian matrix of f evaluated at $\mathbf{a}$.

Question 2. Find the Taylor Polynomial for the given function, the given n, and at the given point $\mathbf{a}$:

- **(a)** $f(x) = \sqrt{x}$, $a = 9$, $n = 4$.
- **(b)** $g(x, y, z) = ye^3x + ze^2y$, $\mathbf{a} = (0, 0, 2)$, $n = 2$.
- **(c)** $h(x, y) = 4x^2 + xy - 3x$, $\mathbf{a} = (1, 1)$, $n = 2$.
- **(d)** $h(x, y) = 4x^2 + xy - 3x$, $\mathbf{a} = (0, 0)$, $n = 53$.

Question 3. For the following functions, calculate the Hessian and the total differential, both in terms of functions and at the point given:

- **(a)** $f(x, y, z) = \frac{z}{\sqrt{xy}}$, $\mathbf{a} = (1, 2, -4)$.
- **(b)** $g(x, y, z) = e^{2x-3y} \sin 5z$, $\mathbf{a} = (0, 0, 0)$.

Question 4. Determine precise conditions on the constant k so that the function

$$h(x, y, z) = kx^2 + kxz - 2yz - y^2 + \frac{k}{2}z^2$$

has a nondegenerate local maximum at the origin. Are there conditions on k where h has a local minimum at the origin?

Question 5. Find the point(s) on the surface $xy + z^2 = 4$ that is (are) closest to the origin. Show that it is (they are) the closest.

Question 6. Locate all critical points, and then any local and global extrema, of the following functions on the regions given:

- **(a)** $f(x, y) = x^2 + xy + y^2 - 6y$ on the rectangle $\big\{(x, y) \bigm| -3 \le x \le 3, 0 \le y \le 5\big\}$.
- **(b)** $g(x, y, z) = x^3 + 3x^2 + e^{y^2+1} + z^2 - 3xz$ on $\mathbb{R}^3$.

Question 7. The Hessian fails to help determine whether the origin is a local extremum of the function $f(x, y, z) = x^2y^3z^4$. Check the Hessian to see that this is true. Then find another way to classify the origin as a local min, max, or neither.

Question 8. For the following functions, use Lagrange multipliers to identify the critical points of f subject to the constraints given.

- **(a)** $f(x, y, z) = xy$, $2x - 3y = 6$.
- **(b)** $f(x, y, z) = x^2 + y^2 + z^2$, $x + y - z = 1$.
- **(c)** $f(x, y, z) = x + y + z$, $y^2 - x^2 = 1$, $x + 2z = 1$.

Question 9. Do the following, taking the care to convince the reader why such an extreme value must exist in each case:

- **(a)** Find three positive numbers whose sum is 18 and whose product is as large as possible.
- **(b)** Find the smallest value of $f(x, y, z) = x + y - z$ on the sphere $x^2 + y^2 + z^2 = 81$.
- **(c)** Find the point farthest from the origin on the ellipse formed by the intersection of the plane $x + y + z = 4$ and the parabolic bowl $z = x^2 + y^2$.