HOMEWORK PROBLEM SET 6: DUE MARCH 13, 2019

AS.110.211 HONORS MULTIVARIABLE CALCULUS
PROFESSOR RICHARD BROWN

Question 1. Prove that the curl of a vector field in $\mathbb{R}^3$ is incompressible. That is, given a C^2-vector field $\mathbf{F} : X \subset \mathbb{R}^3 \to \mathbb{R}^3$, show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Question 2. An important (second-order, partial differential) operator on real-valued functions on $\mathbb{R}^n$ is the *Laplacian Operator*, denoted ∇^2. It is defined by

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

(a) Show that another way to write ∇^2 is $\nabla \cdot \nabla$.

(b) Show that if f and g are C^2-functions on $\mathbb{R}^n$, then

$$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\left(\nabla f \cdot \nabla g\right).$$

(c) Show that $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

Question 3. Demonstrate whether the following is true or false:

(a) The vector field $\mathbf{G} = 2xy\cos z\,\mathbf{i} - y^2\cos z\,\mathbf{j} + e^{xy}\,\mathbf{k}$ is incompressible.

(b) The vector field $\mathbf{G} = 2xy\cos z\,\mathbf{i} - y^2\cos z\,\mathbf{j} + e^{xy}\,\mathbf{k}$ is irrotational.

(c) For any C^2-vector field, if $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F} = \mathbf{0}$.

(d) There exists a C^2-vector field $\mathbf{H}$ on $\mathbb{R}^3$ such that $\nabla \times \mathbf{H} = x\cos^2 y\,\mathbf{i} + 3y\,\mathbf{j} - xyz^2\,\mathbf{k}$.

(e) The vector field $\mathbf{V} = 2x\sin y\cos z\;\mathbf{i} + x^2\cos y\cos z\;\mathbf{j} - x^2\sin y\sin z\;\mathbf{k}$ is the gradient of a C^2-function f on $\mathbb{R}^3$.

(f) If $\mathbf{F}$ and $\mathbf{G}$ are gradient fields, then $\mathbf{F} \times \mathbf{G}$ is incompressible.

Question 4. Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and denote $r = ||\mathbf{r}||$. Show the following:

(a) $\nabla\left(\ln r\right) = \dfrac{\mathbf{r}}{r^2}$.

(b) For $n \in \mathbb{N}$, $\nabla \cdot (r^n\mathbf{r}) = (n+3)r^n$.