HOMEWORK PROBLEM SET 5: DUE MARCH 8, 2019

AS.110.211 HONORS MULTIVARIABLE CALCULUS
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Question 1. Can you solve

$$\begin{aligned} x_2y_2 - x_1 \sin y_1 &= 5 \\ x_2 \sin y_1 + x_1y_2 &= 2 \end{aligned}$$

for y_1, y_2 as functions of x_1, x_2 near the point $(x_1, x_2, y_1, y_2) = (2, 3, \pi, 1)$? What about near the point $(x_1, x_2, y_1, y_2) = \left(0, 2, \frac{\pi}{2}, \frac{5}{2}\right)$?

Question 2. Consider the system:

$$\begin{cases} x_1y_2^2 - 2x_2y_3 = 1 \\ x_1y_1^5 + x_2y_2 - 4y_2y_3 = -9 \\ x_2y_1 + 3x_1y_3^2 = 12 \end{cases} .$$

(a) Show that, near the point $(x_1, x_2, y_1, y_2, y_3) = (1, 0, -1, 1, 2)$, it is possible to solve for y_1, y_2, and y_3 in terms of x_1 and x_2.

(b) From the results of **part (a)**, we may consider y_1, y_2, and y_3 to be functions of x_1 and x_2. Use implicit differentiation and the Chain Rule to evaluate each of $\frac{\partial y_1}{\partial x_1}(1, 0)$, $\frac{\partial y_2}{\partial x_1}(1, 0)$, and $\frac{\partial y_3}{\partial x_1}(1, 0)$.

Question 3. Let $\mathbf{x}(t) \in \mathbb{R}^n$ be a differentiable curve such that $||\mathbf{x}(t)|| = c > 0$, for all t. Show that $\mathbf{x}(t) \cdot \mathbf{x}'(t) = 0$, for every t.

Question 4. For the curve $\mathbf{x}(t) = (t \sin t, t \cos t, t^2)$, do the following:

(a) Calculate the speed, velocity and acceleration of the curve.

(b) Show that the curve, for $t \in [-20, 20]$, lies on the paraboloid S defined by the equation $z = x^2 + y^2$.

Question 5. Do the following:

(a) Sketch the path of $\mathbf{x}(t) = (t, t^3 - 2t + 1)$.

(b) Calculate the line tangent to $\mathbf{x}$ at $t = 2$.

(c) Eliminate the parameter t by writing the curve as a function $y = f(x)$.

(d) Verify by calculation that the line you calculated in **part (b)** is the same line one calculates as the tangent line to the graph of f in **part (c)**.

Question 6. Verify the Product Rule for calculating the velocity of a curve in $\mathbb{R}^3$, given as the cross product of two other curves in $\mathbb{R}^3$. That is, for two differentiable curves $\mathbf{x}, \mathbf{y} : \mathbb{R} \to \mathbb{R}^3$, define $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ and show that

$$\frac{d\mathbf{z}}{dt} = \frac{d\mathbf{x}}{dt} \times \mathbf{y} + \mathbf{x} \times \frac{d\mathbf{y}}{dt}.$$

Question 7. Calculate the length of each of the paths:

(a) $\mathbf{x}(t) = 7\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$, for $1 \leq t \leq 3$.

(b) $\mathbf{y}(s) = \left(\ln s, \frac{s^2}{2}, \sqrt{2}s\right)$, with $s \in [1, 4]$.

Question 8. For the path $\mathbf{x}(t) = (e^{-t}\cos t, e^{-t}\sin t)$, do the following:

(a) Argue that the path spiral in toward the origin at $t \to \infty$.

(b) Show that, for any choice of $a \in \mathbb{R}$, the improper integral

$$\int_a^\infty ||\mathbf{x}'(t)||\ dt$$

converges.

(c) Interpret what the result from **part (b)** says about the path $\mathbf{x}$.

Question 9. Sketch the following vector fields:

(a) $\mathbf{F} = (x, x^2)$.

(b) $\mathbf{G} = (y, -x, 2)$.

Question 10. Show that the vector field $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$ is a gradient vector field. Describe the equipotential surfaces of $\mathbf{F}$ both in words and in sketches.

Question 11. Show that the curve $\mathbf{x}(t) = (\sin t, \cos t, e^{2t})$ is a flow line off the vector field $\mathbf{F} = (y, -x, 2z)$.