HOMEWORK PROBLEM SET 2: DUE FEBRUARY 13, 2019

AS.110.211 HONORS MULTIVARIABLE CALCULUS
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Question 1. Determine whether the set

$$\left\{(x,y) \in \mathbb{R}^2 \;\middle|\; -1 < x < 1\right\} \cup \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x = 2\right\}$$

is open, closed, or neither.

Question 2. Evaluate the following limits, or show that they fail to exist:

(a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{e^x e^y}{x+y+2}$.

(b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2+y^2}$.

(c) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x^2+y^2}}$.

Question 3. Determine whether the functions are continuous on their domain:

(a) $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(b) $g(x,y) = \begin{cases} \frac{x^3+xy^2+2x^2+2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y) = (0,0). \end{cases}$

Question 4. For $f(x,y) = 2x - 10y + 3$, do the following:

(a) Show that if $||(x,y) - (5,1)|| < \delta$, then $|x-5| < \delta$ and $|y-1| < \delta$.

(b) Use the previous part to show that if $||(x,y) - (5,1)|| < \delta$, then $|f(x,y) - 3| < 12\delta$.

(c) Show that $\displaystyle \lim_{(x,y)\to(5,1)} f(x,y) = 3$.

Question 5. Do the steps below to establish that

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0.$$

(a) Show that $|x| \leq ||(x,y)||$, and $|y| \leq ||(x,y)||$.

(b) Show that $|x^3 + y^3| \leq 2(x^2 + y^2)^{3/2}$. (Hint: Begin with the Triangle Inequality and then use part **(a)**.)

(c) Show that if $0 < ||(x, y)|| < \delta$, then $\left|\frac{x^3+y^3}{x^2+y^2}\right| < 2\delta$.

(d) Now prove that $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$.

Question 6. Calculate the partial derivatives of the following:

(a) $f(x, y) = \frac{x^3-y^2}{1+x^2+3y^4}$.

(b) $g(x, y) = \ln\left(\frac{x}{y}\right)$.

(c) $F(x, y, z) = \sin(x^2y^3z^{-4})$.

Question 7. Show that one can rewrite the expression for the derivative of $f : \mathbb{R} \to \mathbb{R}$ at $x = a$, namely

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}, \quad \text{to} \quad \lim_{x\to a} \frac{f(x) - (f(a) + f'(a)(x - a))}{x - a} = 0.$$

This allows us to define the derivative of f at a through the existence of a linear function $h : \mathbb{R} \to \mathbb{R}$, $h(x) = f(a) + f'(a)(x - a)$ which makes the latter limit equation true.