HOMEWORK PROBLEM SET 12: NOT TO BE HANDED IN.

AS.110.211 HONORS MULTIVARIABLE CALCULUS
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Question 1. $\omega = 4\,dx\wedge\, dy - 7\,dy\wedge\, dz$. Evaluate $\omega(\mathbf{v}_1,\mathbf{v}_2)$, where $\mathbf{v}_1 = \mathbf{j} - \mathbf{k}$, and $\mathbf{v}_2 = \mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$.

Question 2. Given $\mathbf{v}_1 = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, and $\mathbf{v}_2 = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, evaluate the following forms on these vectors at the point $\mathbf{p} \in \mathbb{R}^3$:

(a) Find $\omega(\mathbf{v}_1)$, where $\omega = x^2y\,dx + y^2z\,dy + z^3x\,dz$ and $\mathbf{p} = (3,-1,4)$.

(b) Find $\omega(\mathbf{v}_1,\mathbf{v}_2)$, where $\omega = \cos x\,dx\wedge\, dy - \sin z\,dy\wedge\, dz + (y^2+3)\,dx\wedge\, dz$, and $\mathbf{p} = \left(0,-1,\frac{\pi}{2}\right)$.

Question 3. Determine $\omega\wedge\nu$ in the following situations:

(a) On $\mathbb{R}^3$: $\omega = y\,dx - x\,dy$, and $\nu = z\,dx\wedge\, dy + y\,dx\wedge\, dz + x\,dy\wedge\, dz$.

(b) On $\mathbb{R}^4$: $\omega = (x_1+x_2)\,dx_1\wedge\, dx_2\wedge\, dx_3 + (x_3+x_4)\,dx_1\wedge\, dx_2\wedge\, dx_4$, and $\nu = x_1\,dx_1 + 2x_2\,dx_2 + 3x_3\,dx_3$.

Question 4. Evaluate the following:

(a) $\displaystyle\int_C \omega$, where C is the unit circle $x^2+y^2=1$ in the plane, oriented clockwise, and $\omega = y\,dx - x\,dy$.

(b) $\displaystyle\int_{\mathbf{X}} \omega$, where $\mathbf{X} : \mathcal{D} \to \mathbb{R}^3$ is the parameterized helicoid $\mathbf{X}(s,t) = (s\cos t, s\sin t, t)$, defined on $\mathcal{D} = [0,1]\times[0,4\pi]$ and $\omega = z\,dx\wedge\, dy + 3\,dz\wedge\, dx - x\,dy\wedge\, dz$.

(c) $\displaystyle\int_{\mathcal{S}} \nu$, where $\mathcal{S}$ is the portion of the paraboloid $z^2 = x^2+y^2$ with $z\in[0,4]$, oriented with the upward-pointing vector $\mathbf{N}(x,y,z) = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$, and $\nu = e^z\,dx\wedge\, dy + y\,dz\wedge\, dx + x\,dy\wedge\, dz$.

(d) $\displaystyle\int_{\mathbf{X}} \mu$, where $\mathbf{X}(s,t)$, defined on $\mathcal{D} = [1,3]\times[0,2\pi)$, has image in $\mathbb{R}^4$

$$\mathbf{X}(s,t) = (\sqrt{s}\,\cos t, \sqrt{4-s}\,\sin t, \sqrt{s}\,\sin t, \sqrt{4-s}\,\cos t),$$

and $\mu = (x_2^2+x_4^2)\,dx_1\wedge\, dx_3 + (2x_1^2+2x_3^2)\,dx_2\wedge\, dx_4$.

Question 5. For each form ω, calculate its exterior derivative $d\omega$:

(a) $\omega = x_1\,dx_2 - x_2\,dx_1 + x_3x_4\,dx_4 - x_4x_5\,dx_5$.

(b) $\omega = xz\,dx\wedge\, dy - y^2z\,dx\wedge\, dz$.

Question 6. If $\omega = F(x,z)\,dy + G(x,y)\,dz$ is a differential 1-form on $\mathbb{R}^3$, Determine F and G so that $d\omega = z\,dx \wedge\, dy + y\,dx \wedge\, dz$.

Question 7. Verify the generalized Stokes' Theorem for the surface

$$\mathcal{M} = \left\{(x,y,z) \;\middle|\; x^2 + z^2 = 1,\ 0 \leq y \leq 3\right\}$$

and $\omega = z\,dx + (x+y+z)\,dy - x\,dz$.