# HOMEWORK PROBLEM SET 10: DUE APRIL 17, 2019

AS.110.211 HONORS MULTIVARIABLE CALCULUS
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**Question 1.** Do Exercise 1 in **Lecture 18**: Given $\mathcal{D}$ a closed, bounded, elementary region of Type II in the plane, and $\mathbf{F}(x,y) = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ a $C^1$-vector field, show that $\oint_{\partial\mathcal{D}} N\,dy = \iint_{\mathcal{D}} \frac{\partial N}{\partial x}\,dA$.

**Question 2.** Example 18.9 in **Lecture 18** demonstrates that not all irrotational vector fields are conservative. The vector field $\mathbf{F} = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j} + 0\,\mathbf{k}$, defined on $\mathcal{W} = \mathbb{R}^3 - \left\{(0,0,z) \in \mathbb{R}^3 \;\middle|\; z \in \mathbb{R}\right\}$ is one such example. ($\mathcal{W}$ is not simply connected.) To verify, do the following:

- **(a)** Show that $\mathbf{F}$ is irrotational, so show that $\nabla \times \mathbf{F} = \mathbf{0}$.
- **(b)** Show that $\mathbf{F}$ does not have path-independent line integrals by finding a simple closed curve $\mathbf{c}$ where $\int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s} \neq 0$. (Hint: Try the unit circle in the $xy$-plane.)

**Question 3.** Verify Green's Theorem for the following:

- **(a)** $\mathbf{F} = 2y\,\mathbf{i} + x\,\mathbf{j}$, and $\mathcal{D}$ is the semicircular region defined by $y^2 + y^2 \leq a^2$, where $a > 0$, and $y \geq 0$.
- **(b)** $\mathbf{G} = 3xy\,\mathbf{i} + 2x^2\,\mathbf{j}$, and $\mathbf{c}$ is the oriented simple closed curve given by Figure 1.

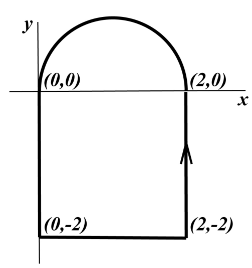


FIGURE 1. The curve $\mathbf{c}$ here encloses a square capped by a semicircle.

**Question 4.** Let $\mathcal{D}$ be a region for which Green's Theorem applies, with its boundary $\partial\mathcal{D}$ oriented as in the theorem. Show that the area of $\mathcal{D}$ is given by either of the following:

$$\mathbf{area}(\mathcal{D}) = \oint_{\partial\mathcal{D}} x\,dy = -\oint_{\partial\mathcal{D}} y\,dx.$$

**Question 5.** For $\mathbf{c}$ any simple closed curve in the plane, show that $\displaystyle\oint_{\mathbf{c}} 3x^2y\,dx + x^3\,dy = 0$.

**Question 6.** Let $f(x,y)$ be *harmonic* (this means that $f$ is $C^2$, with $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$). Then, for any region $\mathcal{D}$ for which Green's Theorem applies, we have

$$\oint_{\partial\mathcal{D}} \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = 0.$$

**Question 7.** Do the following, for $\mathbf{F} = 2xy\,\mathbf{i} + (x^2+z^2)\,\mathbf{j} + 2yz\,\mathbf{k}$:

- **(a)** Calculate the vector line integral of $\mathbf{F}$ along $\mathbf{x}(t) = (t^2, t^3, t^5)$, $0 \le t \le 1$.
- **(b)** Calculate the vector line integral of $\mathbf{F}$ along the line from $(0,0,0)$ to $(1,0,0)$, and then along the line from $(1,0,0)$ to $(1,1,1)$.
- **(c)** Does $\mathbf{F}$ have path-independent line integrals? Explain fully your assessment.

**Question 8.** Determine whether the given vector field is conservative. If so, find a scalar potential for $\mathbf{F}$:

- **(a)** $\mathbf{F} = \dfrac{xy^2}{(1+x^2)^2}\,\mathbf{i} + \dfrac{x^2y}{1+x^2}\,\mathbf{j}$.
- **(b)** $\mathbf{F} = (4xyz^3 - 2xy)\,\mathbf{i} + (2x^2z^3 - x^2 + 2yz)\,\mathbf{j} + (6x^2yz^2 + y^2)\,\mathbf{k}$.

**Question 9.** Find all functions $M(x,y)$ such that the vector field

$$\mathbf{F} = M(x,y)\,\mathbf{i} + (x\sin y - y\cos x)\,\mathbf{j}$$

is conservative.

**Question 10.** Let $\mathcal{S} = \mathbf{X}(\mathcal{D})$ for the map $\mathbf{X}(s,t) = (s^2\cos t, s^2\sin t, s)$, defined on $\mathcal{D} = [-3,3]\times[0,2\pi]$.

- **(a)** Find a normal vector for $\mathcal{S}$ at the point $(s,t) = (-1,0)$.
- **(b)** Determine the tangent plane to $\mathcal{S}$ at the point $(1,0,-1)$.
- **(c)** Find an equation $f(x,y,z) = 0$ whose solutions are precisely $\mathcal{S}$.

**Question 11.** Find the surface area of the following surfaces:

- **(a)** The helicoid: $\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$, for $\mathcal{D} = [0,1]\times[0,4\pi]$.
- **(b)** Part of the paraboloid: The set of points in $\mathbb{R}^3$ satisfying $z = 2x^2 + 2y^2$, for $z \in [2,8]$.

**Question 12.** Represent the lower hemisphere, including the equatorial circle, defined by the equation $x^2+y^2+z^2 = 9$ as a piecewise smooth parameterized surface.