HOMEWORK PROBLEM SET 1: DUE FEBRUARY 6, 2019

AS.110.211 HONORS MULTIVARIABLE CALCULUS
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Question 1. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x, y) = 2x^2 + y^2 - 7$.

(a) Determine the domain and range of g and show that g is neither 1-1 nor onto.

(b) Restrict the domain of g to a new function with the same rule of assignment and the same range, but which is injective.

(c) Restrict the codomain of g to a new function with the same rule of assignment and the same domain, but which is surjective.

Question 2. Find the domain and range of the following:

(a) $f(x, y) = \ln(x + y)$.

(b) $g(x, y, z) = \frac{1}{\sqrt{4-x^2-y^2-z^2}}$.

Question 3. Consider a mapping that takes every nonzero vector $\mathbf{x} \in \mathbb{R}^3$ to the vector of length 4 that points in the opposite direction of $\mathbf{x}$. Write an expression for this mapping, both symbolically (determining the domain and range), and via an expression (including the component functions).

Question 4. Consider the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & -1 \\ 5 & 0 \\ -6 & 3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(a) Write out the component functions of $\mathbf{f}$ in terms of x_1 and x_2.

(b) Describe the range of $\mathbf{f}$.

Question 5. For the following functions, describe the shape and position of several level sets (curves in these cases). Then use this information to sketch the graphs. (Feel free to use a computer.)

(a) $f(x, y) = x^2 + y^2 - 9$.

(b) $f(x, y) = 4x^2 + 9y^3$.

(c) $f(x,y) = \dfrac{y}{x}$.

(d) $f(x,y) = 3 - 2x - y$.

Question 6. Describe the graph of $g(x,y,z) = xy - yz$ by sketching a few level surfaces.

Question 7. Do the following:

(a) Describe the graph of $g(x,y,z) = x^2 + y^2$ by computing a few level surfaces.

(b) For any function $g(x,y,z)$ that involves only x and y in its expression, what can you say about the level surfaces of g.

(c) For any function $g(x,y,z)$ that involves only x and z in its expression, what can you say about the level surfaces of g.

(d) For any function $g(x,y,z)$ that involves only x in its expression, what can you say about the level surfaces of g.

Question 8. Consider the surface given as the graph of the equation $x^2 - 2 = xz - xy$ in $\mathbb{R}^3$.

(a) Find a function $F : \mathbb{R}^3 \to \mathbb{R}$ so that the surface comprises a level set of F.

(b) Find a function $f(x,y)$ so that the surface may be considered the graph of $z = f(x,y)$.

Question 9. Read up on the notion of Stereographic Projection by looking it up on Wikipedia here: `https://en.wikipedia.org/wiki/Stereographic_projection`. You will notice that this does not match our definition of projection, given in the lecture notes (indeed, the domain and range are not the same). Make it fit the definition by creating a domain $X \in \mathbb{R}^3$, so that the map takes X to X and satisfies the definition. Note that the projection given on Wikipedia takes the unit 2-sphere

$$S^2 = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 = 1\right\}$$

to the xy-plane via the map

$$(x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$