February 20, 2019

LECTURE 9: CURVES IN REAL SPACE.

110.211 HONORS MULTIVARIABLE CALCULUS
PROFESSOR RICHARD BROWN

Synopsis. Today we begin the study of Chapter 3 on vector-valued functions. For the most part, there are only two topics of discussion here: paths or curves and vector fields, respectively defined as functions from the real line into n-space, or functions from n-space into itself. The reason for an entire chapter on these two items is that they play a huge role in a solid general understanding of all of the calculus of vector-valued functions of more than one variable. They also introduce the idea of a geometric object begin completely defined by a function, allowing us to fold geometry into the analysis of functions in a fundamental way. This is one of the core principles of higher mathematics. Today, curves in n-space and some of their properties. One defining characteristic of a curve in n-space is that its length should be independent of its parameterization, even though we calculate the length using the parameterization. This extra document details why this is so:

Helpful Documents. PDF: `ParameterizationIndependence`.

Curves in $\mathbb{R}^n$. We start with a definition:

Definition 9.1. A *curve* or *path* in $\mathbb{R}^n$ is a continuous function $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{R}^n$, where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, defined on an interval.

Note that the image of $\mathbf{x}(t)$ is an n-vector for each value of $t \in I$. If $\mathbf{x}$ is differentiable as a function, then its derivative is also an n-vector, and

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{x}'(t) = \begin{bmatrix} \frac{dx_1}{dt}(t) \\ \vdots \\ \frac{dx_n}{dt}(t) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

We sometimes call the derivative vector the *velocity*, and denote it $\mathbf{v}(t) = \mathbf{x}'(t)$.

At a point $t_0 \in I$, $\mathbf{v}(t_0)$ is represented by a vector, based at $\mathbf{x}(t_0)$ and tangent to the curve $\mathbf{image}(\mathbf{x})$ Here, we simply denote the entire curve as $\mathbf{x}$. Now, as long as $\mathbf{v}(t_0) \neq \mathbf{0}$, this vector defines a unique tangent line to $\mathbf{x}$ at $t = t_0$, parameterized as

$$\ell(s) = \mathbf{x}(t_0) + s\mathbf{v}(t_0), \quad \text{or}$$
$$\ell(s) = \mathbf{x}(t_0) + (t - t_0)\mathbf{v}(t_0), \quad \text{for} \quad s = t - t_0.$$

Note that the line $\ell = \mathbf{span}\,\{\mathbf{v}(t_0)\}$.

Here, the *speed* of $\mathbf{x}(t)$ at $t = t_0$ is simply the size of the velocity vector at t_0, so $||\mathbf{v}(t_0)||$. The interpretation is of a bead moving along a piece of wire that is the curve. The bead is at $\mathbf{x}(t_0)$ at time $t = t_0$ and moving with (instantaneous) speed $||\mathbf{v}(t_0)||$ then. All of this is a topic of a standard single variable calculus course, since all of the derivatives here are calculated according to the component functions $x_i : I \to \mathbb{R}$, each of which is a real-valued on $I \subset \mathbb{R}$.

Indeed, let $x = f(t)$ and $y = g(t)$, for $t \in I \subset \mathbb{R}$, define a parametric curve in $\mathbb{R}^2$. If $f, g \in C^1$, then $\frac{dx}{dt} = f'(t)$ and $\frac{dy}{dt} = g'(t)$, and when defined,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

defines the tangent line in $\mathbb{R}^2$ to the curve at (x_0, y_0):

$$y = \left(\left. \frac{dy}{dx} \right|_{(x_0,y_0)} \right) (x - x_0) + y_0, \quad x_0 = f(t_0), \text{ and } y_0 = g(t_0).$$

This construction was useful for studying curves that are defined only implicitly and not representable as functions $y(x)$ or $x(y)$: If a curve is defined as $F(x, y) = 0$, then we can calculate $\frac{dy}{dx}$ in two ways: (1) implicitly, or (2) via a parameterization like above. But we can use the language of vector calculus, now, to revisit these methods:

Implicit differentiation. Assume that $y = y(x)$ is an implicit function of x. The the equation $F(x, y) = 0$ looks like $F(x, y(x)) = 0$, and is only a function of x. Thus we can differentiate with respect to x and get

$$\frac{d}{dx} F(x, y(x)) = \frac{\partial}{\partial x} F(x, y) + \frac{\partial}{\partial y} F(x, y) \frac{dy}{dx} = 0.$$

Thus, we get

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

Via parameterization. Both $x = x(t)$ and $y = y(t)$ are functions of t, so $F(x, y) = F(x(t), y(t)) = 0$, adn F is only a function of t. Thus

$$\frac{d}{dt} F(x(t), y(t)) = F_x(x, y) \frac{dx}{dt} + F_y(x, y) \frac{dy}{dt} = 0.$$

This implies again the SAME result:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

Thinking of a curve as a function affords us all of the tools of calculus to study the geometry of curves:

(1) We can attribute higher derivatives to geometric features like acceleration,

$$\mathbf{a}(t) = \frac{d}{dt} \mathbf{v}(t) = \frac{d^2}{dt^2} \mathbf{x}(t),$$

and jerk, etc.

(2) We can recover quantities like distance via integrating velocity, so that

$$\mathbf{x}(t) = \int_{t_0}^{t} \mathbf{v}(s)\, ds.$$

Do keep in mind, though, that integrating a vector means integrating each component, noting that the constant of integration is, again, a vector.

(3) Derivative rules, again, behave well with respect to curves. For example, the Product Rule and the Dot Product:

$$\frac{d}{dt}\left[\mathbf{x} \cdot \mathbf{y}(t)\right] = \frac{d\mathbf{x}}{dt} \cdot \mathbf{y}(t) + \mathbf{x}(t) \cdot \frac{d\mathbf{y}}{dt}.$$

(4) Facilitates geometric study:

Example 9.2. If $\mathbf{x}(t) \subset \mathbb{R}^n$ is a C^1-curve, with $||\mathbf{x}(t)|| = c > 0$, for all $t \in I$, then $\mathbf{x}'(t) \cdot \mathbf{x}(t) = 0$, for every $t \in I$.

Exercise 1. Prove this result.

Recall from Calculua II, for $f : [\alpha, \beta] \to \mathbb{R}$, the length of $\mathbf{graph(f)} \subset \mathbb{R}^2$ on $[\alpha, \beta]$ is

$$L = \int_{\alpha}^{\beta} \sqrt{1 + (f(x))^2}\, dx,$$

or if the curve is a parametric curve,

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Exercise 2. Show that these two quantities are the same.

This last formula is tailor-made for us: Let $\mathbf{x} : [a, b] \to \mathbb{R}^2$, $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Here, given a partition on the interval $[a, b]$,

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b,$$

one looks for the approximate length of the curve on the subinterval $[t_{i-1}, t_i]$, and then adds up the approximations on each subinterval to get an approximation of the length of the curve. For each subinterval $[t_{i-1}, t_i]$, calculate Δt_i. Now approximate the length of the curve on a subinterval by using Euclidean distance between $\mathbf{x}(t_{i-1}$ and $\mathbf{x}(t_i)$. The approximate length of the curve in the ith subinterval is

$$||\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})|| = \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}.$$

But we can write

$$x(t_i) - x(t_{i-1}) = \Delta x_i = x'(t_i^*)\Delta t_i$$
$$y(t_i) - y(t_{i-1}) = \Delta y_i = y'(t_i^{**})\Delta t_i$$

by the Mean Value Theorem for some t_i^* and t_i^{**} in $[t_{i-1}, t_i]$. So the approximate length of the curve, given the partition, is

$$\text{approx } L = \sum_{i=1}^{n} \sqrt{(x'(t_i^*))^2 + (y'(t_i^{**}))^2}\, \Delta t_i.$$

And the actual length is found by taking the limit as the largest $\Delta t_i \to 0$:

$$\begin{aligned} L &= \lim_{\max_i \Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{\left(x'(t_i^*)\right)^2 + \left(y'(t_i^{**})\right)^2}\ \Delta t_i \\ &= \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\ dt \\ &= \int_a^b ||\mathbf{x}'(t)||\ dt, \end{aligned}$$

where the quantity $||\mathbf{x}'(t)||$ is the size of the velocity vector at time t, otherwise known as the *speed* of the curve at t. All of this works in $\mathbb{R}^n$, $n \in \mathbb{N}$.

Definition 9.3. The length of $\mathbf{x} : [a, b] \to \mathbb{R}^n$, a C^1-parameterized curve in $\mathbb{R}^n$ is

$$L(\mathbf{x}) = \int_a^b ||\mathbf{x}'(t)||\ dt.$$

Some notes:

- One integrates speed to recover distance (length traveled)!
- Even if the curve is only piecewise C^1 (so maybe it has corners), this still works, as integrals are additive.
- This formula seems to critically depend on the parameterization. But it does not! (See the Helpful document for a proof.) TO verify, reparameterize and reintegrate. or better yet, parameterize *intrinsically*, using length itself as the parameter on the curve.

Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a path with non-zero velocity everywhere (so that $\mathbf{v}(t) \neq \mathbf{0}$, $\forall t \in [a, b]$). Denote by $\mathbf{p}_0 = \mathbf{x}(a)$, and $\mathbf{p} = \mathbf{x}(s)$, where

$$s(t) = \int_a^t ||\mathbf{x}'(\tau)||\ d\tau.$$

(Note that the use of τ is simply a dummy variable in a definite integral, and never actually appears on the curve.) Some things to think about:

- Since $\mathbf{x}'(t) \neq \mathbf{0}$, the length is always positive, and $s(t)$ is a strictly increasing function. As such, it is invertible, and we can reparameterize $\mathbf{x}(t)$ to
$$\mathbf{x}(s) = \mathbf{x}\left(t(s)\right)$$
as a function of s.
- In practice, $t(s)$ may be difficult of near impossible to find, but the total length of the curve is
$$s(b) = \int_a^t ||\mathbf{x}'(\tau)||\ d\tau = \int_a^t ||\mathbf{x}'(t)||\ dt,$$
which is just the length of the curve in the t parameter. Hence reparameterization does not change length.
- $s(t)$ is c^1 when $\mathbf{x}$ is, and
$$s'(t) = \frac{ds}{dt} = \frac{d}{dt}\left[\int_a^t ||\mathbf{x}'(\tau)||\ d\tau\right] = ||\mathbf{x}'(t)||\,.$$

So under this reparameterization, the derivative is just the spread of the curve at the old value of t.

So we can use this to calculate the tangent vector in the new parameter: Write $\mathbf{x}(t) = \mathbf{x}(s(t))$. The differentiate, using the Chain Rule:

$$\mathbf{x}'(t) = \frac{d}{dt}\mathbf{x}(s(t)) = \mathbf{x}'(s) \cdot s'(t) = \mathbf{x}(s) \left|\left|\mathbf{x}'(t)\right|\right|, \quad \text{so} \quad \mathbf{x}'(s) = \frac{\mathbf{x}'(t)}{\left|\left|\mathbf{x}'(t)\right|\right|}.$$

Conclusions?

(1) In the new parameter, the arclength traverses the curve at unit speed always.
(2) $\mathbf{x}'(s)$ is just the *normalization* of the tangent vector at the same point as $\mathbf{x}(t)$.

Definition 9.4. For a C^1-path $\mathbf{x} : [a, b] \to \mathbb{R}^n$, the *unit tangent vector* to $\mathbf{x}$ at $t = t_0$ is

$$\mathbf{T}(t_0) = \frac{\mathbf{x}'(t_0)}{\left|\left|\mathbf{x}'(t_0)\right|\right|},$$

and is just the normalized velocity.

This concept of a normalized velocity vector will be very important later on in the course.