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LECTURE 8: IMPLICIT AND INVERSE FUNCTION THEOREMS.

110.211 HONORS MULTIVARIABLE CALCULUS
PROFESSOR RICHARD BROWN

Synopsis. Here, give a treatment of both the Implicit Function Theorem (for real-valued functions), and the Inverse Function Theorem. These are very powerful theorems that expose some of the hidden structure of real-valued and vector-valued functions of more than one variable. We will study the ideas in class, and here is a proof of the Implicit Function Theorem for a function on (a subset of) three space. And here is a Mathematica Notebook for this class.

Helpful Documents.

- Mathematica: `ImplicitFunctionTheoremExample`.
- PDF: `IFTproof`

The Implicit Function Theorem.

8.0.1. *In three variables.* Recall the definition of a c-level set of a function $F : X \subset \mathbb{R}^n \to \mathbb{R}$:

$$S_c = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; f(\mathbf{x}) = c \right\}.$$

For this discussion, let $F \in C^1$, and $n = 3$. We are using an upper case F here for a reason, which should be clear in the following discussion. Here is an example for motivation:

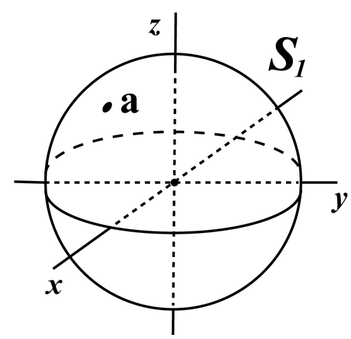

FIGURE 8.1. The unit 2-sphere in $\mathbb{R}^3$, defined as S_1.

Example 8.1. Define $F : \mathbb{R}^3 \to \mathbb{R}$ by $F(x, y, z) = x^2 + y^2 + z^2$, and let $\mathbf{a} \in S_1$. Geometrically, here, S_1 is the unit sphere in $\mathbb{R}^3$, depicted in Figure 8.1. Some questions:

Question 1. Is it possible to view S_1 as the graph of a function where we think of one variable as a dependent variable and all of the others still independent. Thus, in this case, can we write S_1 as the graph of $z = f(x, y)$ (this would be a different function than F)? The answer here is no! But, specifically, why not?

Question 2. Is it possible to write S_1 as $z = f(x, y)$ "locally", near $\mathbf{a} \in S_1$? The answer here is "depends...". But specifically, depends on what? Where $\mathbf{a}$ is located. Specifically whether the point in question is along the equator or not.

Question 3. So what information about F can be used to determine whether we can locally think of a level set of a function as the graph of (a different) function, with one variable a dependent variable and the other independent?

A central tool for this study will be the gradient of F: $\nabla F(\mathbf{a}) = \begin{bmatrix} F_x(\mathbf{a}) \\ F_y(\mathbf{a}) \\ F_z(\mathbf{a}) \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$. For $F(x,y,z) = x^2+y^2+z^2$, then, what property does ∇F have for points along the equator that is different from other points on S_1? Namely, for points like $\mathbf{b}$ along the equator in Figure 8.2, $F_z(\mathbf{b}) = 0$. Off of the equator, like point $\mathbf{a}$ in Figure 8.2, $F(\mathbf{a}) \neq 0$. So for $\mathbf{b} \in S_1$ along the equator, take any small open set *inside* S_1 containing $\mathbf{b}$ (This is the dotted oval sitting on the surface, where it bends around the sphere a bit). If we try to write these points in a form $z = f(x,y)$, we would wind up with some values for x and y with two points for z, following the function $z = \pm\sqrt{1-x^2-y^2}$. Here, $F_z(\mathbf{b}) = 0$ means that the gradient vector has no component in the z direction. It means that the gradient vector is "horizontal" (read: perpendicular to the z-direction). This means that the tangent plane to S_1 at the point $\mathbf{b}$ would look "vertical" here (all vector with only a z-component would be inside the tangent plane).

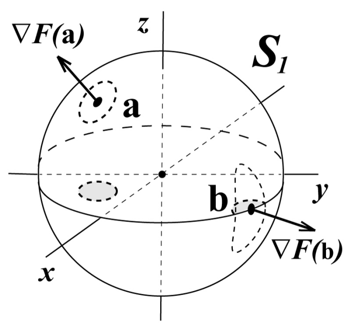

FIGURE 8.2. ∇F helps determine where level sets locally look like graphs of functions.

Hence the condition that $F_z(\mathbf{a}) \neq 0$ is a sufficient condition for being able to locally write $F(x,y,z) = c$ near $\mathbf{a}$ as $z = f(x,y)$ for some function f. This works equally well in n-dimensions:

Theorem 8.2 (Theorem 2.6.5)**.** *Let* $F : X \subset \mathbb{R}^n \to \mathbb{R}$ *be* C^1 *and* $\mathbf{a} = (a_1, \ldots, a_{n-1}) \in S_c$*, where*

$$S_c = \left\{ \mathbf{x} \in X \;\middle|\; F(\mathbf{x}) = c \right\}.$$

If $F_{x_n}(\mathbf{a}) \neq 0$*, then there exists a neighborhood* U *of* $(a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}$*, a neighborhood* V *of* $a_n \in \mathbb{R}$*, and a* C_1*-function* $f : U \subset \mathbb{R}^{n-1} \to V$*, such that when* $(x_1, \ldots, x_{n-1}) \in U$*, and* $x_n \in V$*, then* $x_n = f(x_1, \ldots, x_n)$*.*

Go back to our example of the 1-level set S_1 of the function $F(x,y,z) = x^2+y^2+z^2$. If we choose $\mathbf{a}$ strictly inside the northern hemisphere of S_1, as in Figure 8.3, then for these points, we can "solve" for z as a function of x and y:

$$z = f(x,y) = \sqrt{1-x^2-y^2}.$$

But to do this on a neighborhood of $U(\mathbf{a})$, we need to make sure that U includes no points from the equator. So choose $\mathbf{a} \subset S_1$ from the northern hemisphere. Now since $\mathbf{a} = (a_1, a_2, a_3)$ satisfies $a_1^2+a_2^2+a_3^2 = 1$, and $a_3 > 0$, it follows that $a_1^2+a_2^2 < 1$, so that (a_1, a_2), in the xy-plane, is inside the unit circle there. The distance between (a_1, a_2) and the unit circle in the xy-plane is $1 - (a_1^2+a_2^2) > 0$, so choose $\delta = \frac{1}{2}\left(1 - (a_1^2+a_2^2)\right)$. Then the neighborhood $U(a_1,a_2) = B_\delta(a_1,a_2)$ lies completely inside the unit circle in the xy-plane (See Figure 8.3). Take $V \in S_1$, where $V = f(U)$, and the theorem holds.

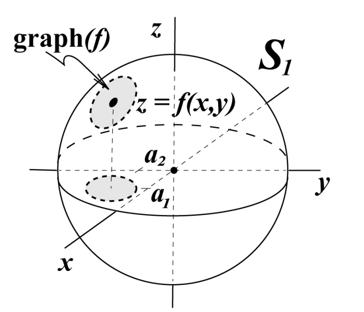

FIGURE 8.3. Near $\mathbf{a}$, S_1 looks like the graph of $z = f(x,y) = \sqrt{1-x^2-y^2}$.

Example 8.3. Let $G(x,y,z) = 2xy^2+xyz-2z^2$, and $\mathbf{a} = (2,-3,3)$. Can we write $z = f(x,y)$ near $\mathbf{a}$? In essense, this is a question of when it is possible to "solve" for z in terms of x and y. In practice, this theorem and idea provides the ability to solve for one variable in terms for the others even in the case where algebraically, it is extremely difficult or not possible.

For G, we can answer this question quickly: Since

$$G(\mathbf{a}) = G_z(2,-3,3) = (xy-4z)\Big|_{(2,-3,3)} = (2)(-3)-4(3) = -18 \neq 0,$$

the answer is yes!. Basically, since the z-component of the gradient is not 0 at $\mathbf{a}$, it will remain not 0 at all points *near* $\mathbf{a}$. Thus the gradient vector will not be horizontal near $\mathbf{a}$ and the tangent planes to the level-sets of G containing the nearby points will still not be vertical.

To continue with this example, at the point $\mathbf{b} = (0,4,0)$, we have

$$\nabla G(\mathbf{b}) = \begin{bmatrix} G_x(\mathbf{b}) \\ G_y(\mathbf{b}) \\ G_z(\mathbf{b}) \end{bmatrix} = \begin{bmatrix} 2y^2+yz \\ 4xy+xz \\ xy-4z \end{bmatrix}\Bigg|_{(0,4,0)} = \begin{bmatrix} 32 \\ 0 \\ 0 \end{bmatrix}.$$

Due to this, we cannot write z as a function of x and y, near $\mathbf{b}$. We also cannot write y as a function of x and z there. However, we can find a function (at least in theory) so that $x = g(y,z)$, near $\mathbf{b}$.

We can directly calculate the tangent plane to the level set of G near the points $\mathbf{a}$ and $\mathbf{a}$, again using the gradient, in any case that the gradient has at least one component that is not 0. Here,

$$\nabla G(\mathbf{a}) = \begin{bmatrix} G_x(\mathbf{a}) \\ G_y(\mathbf{a}) \\ G_z(\mathbf{a}) \end{bmatrix} = \begin{bmatrix} 2y^2+yz \\ 4xy+xz \\ xy-4z \end{bmatrix}\Bigg|_{(2,-3,3)} = \begin{bmatrix} 9 \\ -18 \\ -18 \end{bmatrix}.$$

Then the equation of the tangent plane is

$$\begin{aligned} \nabla G(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a}) &= 0 \\ 9(x-2) - 18(y+3) - 18(z-3) &= 0 \\ z &= -1 + \frac{1}{2}x - y. \end{aligned}$$

Note that $\mathbf{a}$ is actually inside this tangent plane.

For $\mathbf{b} = (0,4,0)$, we have $\nabla G(\mathbf{b}) = \begin{bmatrix} 32 \\ 0 \\ 0 \end{bmatrix}$, so

$$\nabla G(\mathbf{b}) \cdot (\mathbf{x}-\mathbf{b}) = 0 = 32(x-0) + 0(y-4) + 0(z-0) = 32x.$$

But this is simply the plane defined by the equation $x = 0$, or the yz-plane in $\mathbb{R}^3$. Again, note that $\mathbf{b}$ is inside the yz-plane.

Note: There is a general version of the Implicit Function Theorem for vector-valued functions, but for now, we will move on to a related idea:

8.1. **The Inverse Function Theorem.** Here is a question: Let $y = f(x) = e^x$. Does $f(x)$ have an inverse? This question is really an existence question. One could answer it by actually constructing an inverse function. One can also answer it by appealing to the fact that e^x is injective, and noting that injective functions do have inverses. Specifically, the function here is $f : \mathbb{R} \to \mathbb{R}$, $y = f(x) = e^x$, but

$$\textbf{image}(f) = \mathbb{R}_+ = \left\{ x \in \mathbb{R} \;\middle|\; x > 0 \right\}.$$

Hence only if we restrict the codomain of f to $\mathbb{R}_+$, can we actually construct the inverse: For $f : \mathbb{R} \to \mathbb{R}_+$, $f(x) = e^x$, construct $g : \mathbb{R}_+ \to \mathbb{R}$, $g(x) = \ln x$. Then one can show that $(f \circ g)(x) = x$ on $\mathbb{R}_+$ and $(g \circ f)(x) = x$ on $\mathbb{R}$.

In practice, at times, one would show that an inverse exists by simply taking $y = f(x)$, and attempting to "solve for x". Or one could graph the function and look to see that it satisfies the "horizontal line test", a graphical tool for establishing injectivity, since if a function satisfies the horizontal line test, then its inverse will satisfy the vertical line test, thus verifying that the inverse is actually a function. Without these tools, sometimes it is necessary to know if a function has an inverse even if the expression is not necessary. For example, does $y = x^2 + 5\cos x - e^x$ have an inverse? Does it have one on $[0, 1]$? Without other aids, graph this function to see.

In vector calculus, these questions become much more complicated (try graphing a non-linear function from three space to three space), even as the ideas behind them are precisely the same. Suppose $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{a} \in X \subset \mathbb{R}^n$ open, and $\mathbf{f} \in C^1$. If $\det D\mathbf{f}(\mathbf{a}) \neq 0$, then $\exists U \subset X$, and open neighborhood, where (1) $\mathbf{f}\big|_U$ is 1-1, (2) $\mathbf{f}(U) = V$ is open in $\mathbb{R}^n$, and (3) a uniquely defined inverse function $\mathbf{g} : V \to U$, $g \in C^1$, where

$$(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \mathbf{x} \quad \text{and} \quad (\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{x}.$$

We say $\mathbf{f}$ and $\mathbf{g}$ are inverses of each other, and write $\mathbf{f}^{-1} = \mathbf{g}$ and $\mathbf{g}^{-1} = \mathbf{f}$.

Notes:

- Given $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}^n$, then $\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \mathbf{y}$ is a system of n (nonlinear) equations, each writing a dependent variable y_i as a function of the n independent variables $x_1, \ldots, x_n$. Can we solve this system for the x-variables, writing each of them as a function of the variable $y_1, \ldots, y_n$? In essence, can we rewrite the system as
$$\mathbf{x} = \mathbf{g}(\mathbf{y}),$$
thereby finding the inverse function, where $\mathbf{g} = \mathbf{f}^{-1}$, at least locally to a point $\mathbf{a}$? The answer to this question is yes, but only if $\det D\mathbf{f}(\mathbf{a}) \neq 0$.
- If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then $\mathbf{f}(\mathbf{x}) = A_{n \times n}\mathbf{x} = \mathbf{y}$, for some square matrix A. The question is: Is it possible to find a new matrix A^{-1} so that $\mathbf{x} = A^{-1}\mathbf{y}$? Again, the answer is yes, but only if $\det A \neq 0$.

The Inverse Function Theorem is simply the nonlinear (local) version of this!

Example 8.4. Is it possible to solve $u = xy$, $v = x - y$ for x and y as functions of u and v near the point $\mathbf{a} = (1, 1)$ in the plane? How about near the point $\mathbf{b} = (-1, 1)$? Answer these questions, and where one can invert the system, do so.

To set up this problem, let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbf{f}(x, y) = (xy, x - y)$. The strategy here will be to calculate the derivative of $\mathbf{f}$, evaluated at $\mathbf{a}$ and $\mathbf{b}$, and see if its determinant is non-zero. Where it is non-zero, invert the system.

Here,

$$D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} y & x \\ 1 & -1 \end{bmatrix}, \quad \text{so} \quad D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad D\mathbf{f}(\mathbf{b}) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

It is easy to see that $\det D\mathbf{f}(\mathbf{a}) = -1 \neq 0$, while $\det D\mathbf{f}(\mathbf{b}) = 0$. Hence the system is invertible near $\mathbf{a}$ but not near $\mathbf{b}$.

To invert the system, write $v = x - y$ as $x = v + y$, and then

$$u = xy = (v+y)y = vy + y^2.$$

Solving this for y, we get $y = \frac{-v \pm \sqrt{v^2+4u}}{2}$. But without knowing which sign to choose, this is not yet a function. We then note here that when $(x, y) = (1, 1)$, then $(u, v) = (1, 0)$. Hence the plus sign in the y expression is the one that is compatible to this, since when $u = 1$ and $v = 0$, y must equal 1. Hence we get the system

$$\begin{aligned} x &= v + \frac{-v + \sqrt{v^2 + 4u}}{2} = \frac{v + \sqrt{v^2 + 4u}}{2} \\ y &= \frac{-v + \sqrt{v^2 + 4u}}{2}. \end{aligned}$$

Finally, note that at $\mathbf{b}$, $x = -1$ and $y = 1$. This makes $u = -1$ and $v = -2$. Now, can you see why x and y cannot be functions of u and v near $\mathbf{b}$?