February 4, 2019

LECTURE 3: LIMITS.

110.211 HONORS MULTIVARIABLE CALCULUS
PROFESSOR RICHARD BROWN

Synopsis. Today, we define and investigate the notion of a limit in more than one dimension. This is much more subtle than in the Calculus I case, and much harder to fully investigate using the definition alone. Fortunately, all of the "nice" functions from Calculus I are still "nice" in their multivariable generalization. Also, all of the properties of limits developed in single variable calculus are still valid. We will not go deep in this section, but just survey some ideas which we will explore in more detail in the context of more advanced material. The accompanying Mathematica document details some of the more basic pathological functions, where limits do not exist even as intuition indicates they should.

Helpful Documents.

- Mathematica: `PlottingSurfaces`, and
- PDF: `ProductRule`.

Limits. Recall from Calculus I the definition of a limit of a function at a point:

Definition 3.1. Let $I \subset \mathbb{R}$ be open and $f : I \to \mathbb{R}$ a real-valued function on I. Then f has a *limit* L at $x = c \in I$, denoted

$$\lim_{x \to c} f(x) = L,$$

if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

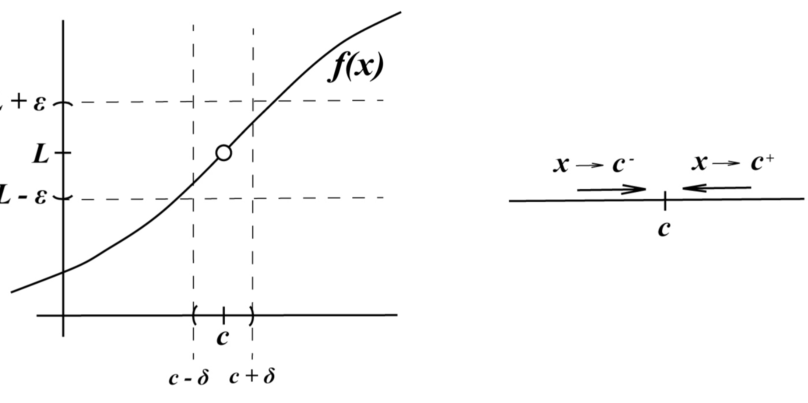

FIGURE 3.1. On the left, $f(x)$ has the limit L at $x = c$. On the right, x can only approach c in $\mathbb{R}$ form two directions.

Some notes:

- Defining a limit *at* c gives us a notion of what happens to the function *near* c. This is the essence of what calculus is really all about!

- Limits are not concerned at all with what happens *at c*.
- If, anytime one can define a small (ϵ-)interval around L, one can find a small (δ-)interval of inputs (around c, but not necessarily at c) all of whose function values stay in the function-value interval (see the left side of Figure 3.1), then near c, all function values stay near L, and the limit will exist.
- Of course, limits can exist even when function values at c are different or nonexistent. In Figure **??**, the limit is the same at $x = c$ for all three graphs.
- In $\mathbb{R}$, the idea of x approaching c involves only 2 possible directions, as shown on the right of Figure 3.1. These correspond to the one-sided limits

$$\lim_{x \to c^-} f(x), \quad \text{and} \quad \lim_{x \to c^+} f(x).$$

  And only when these two "side" limits both exist and are equal, does the actual limit exist.

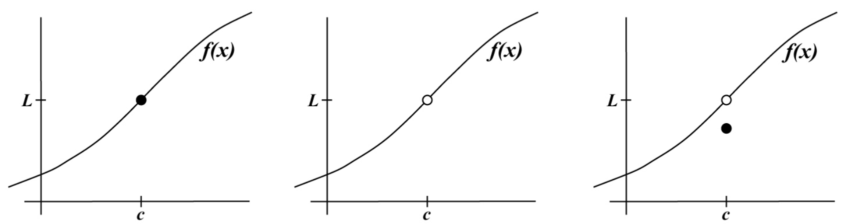

FIGURE 3.2. In all three cases here, $\lim_{x \to c} f(x) = L$.

In Figure 3.3, at right, $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$. Hence $\lim_{x \to c} f(x)$ does not exist. To see this, make a choice for what the limit L could possibly be. Then choose an $\epsilon > 0$ which is small enough to not include both ends of $f(x)$ near $x = c$. Then there will always be points x arbitrarily close to c where $f(x) \notin (L - \epsilon, L + \epsilon)$.

In the case of a vector-valued function of more than one variable, $\mathbf{f} : X \in \mathbb{R}^n \to \mathbb{R}^m$ with $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$, we seek to construct a proper definition for a limit in this new case:

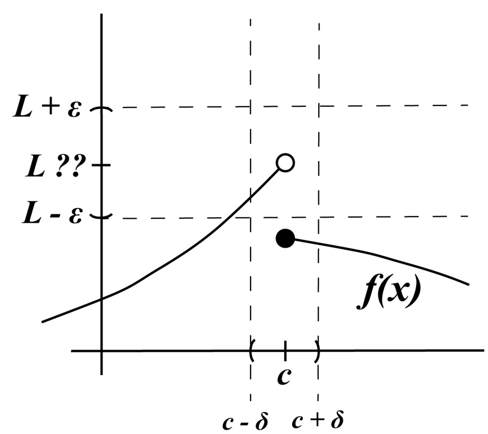

FIGURE 3.3. What possible choice for L could work as a limit for $f(x)$ at $x = c$?

Definition 3.2. $\mathbf{f}$ has a limit $\mathbf{L}$ at $\mathbf{x} = \mathbf{c}$, denoted

$$\lim_{\mathbf{x} \to \mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

if, for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < ||\mathbf{x} - \mathbf{c}|| < \delta$, then $||\mathbf{f}(\mathbf{x}) - \mathbf{L}|| < \epsilon$.

Here, $||\cdot||$ is the Euclidean norm in real space, defined by

$$||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Notice all of the similarities, and one big difference: The number of ways to approach $\mathbf{c}$ in the domain makes things a lot more complicated! See Figure 3.4

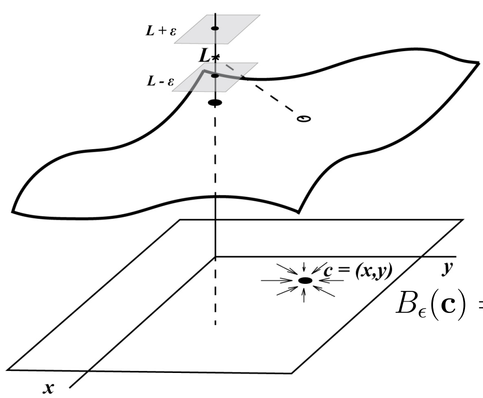

FIGURE 3.4. Approaching a domain point c in two dimensions.

To understand this, we need to introduce some topology: The Euclidean metric on $\mathbb{R}^n$ allows for a nice definition of an "open" set, much like an open interval in $\mathbb{R}$.

3.1. Topology in $\mathbb{R}^n$.

Definition 3.3. An *open ball* of radius $\epsilon > 0$, centered at $\mathbf{c} \in \mathbb{R}^n$ is

$$B_\epsilon(\mathbf{c}) = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; ||\mathbf{x} - \mathbf{c}|| < \epsilon \right\}.$$

Some notes:

- In $\mathbb{R}^3$, this is the usual ball you played with as a kid (see Figure 3.5), but without the skin!
- In $\mathbb{R}^2$, it is the disk of radius ϵ without the circle edge. And in $\mathbb{R}$? How about $\mathbb{R}^17$?
- One can think of this ball as the set of all vectors of length less than ϵ based at $\mathbf{c}$ (and not at $\mathbf{0}$.

Definition 3.4. A *closed ball* of radius $\epsilon > 0$, centered at $\mathbf{c} \in \mathbb{R}^n$ is

$$\overline{B}_\epsilon(\mathbf{c}) = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; ||\mathbf{x} - \mathbf{c}|| \le \epsilon \right\}.$$

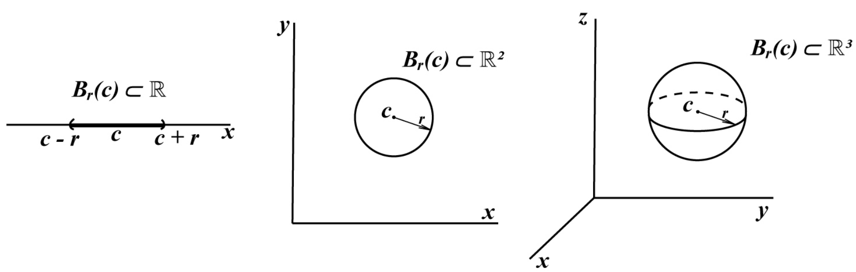

FIGURE 3.5. An r-ball about $c \in \mathbb{R}$, $c \in \mathbb{R}^2$ and $c \in \mathbb{R}^3$.

Here, the "skin" of the ball (technically, its *boundary*, is the set $\left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; ||\mathbf{x} - \mathbf{c}|| = \epsilon \right\}$. What does this skin look like in $\mathbb{R}^5$, for example? In $\mathbb{R}$? Does this thing have a name? It winds up being the boundary of both $B_\epsilon(\mathbf{c})$ and $\overline{B}_\epsilon(\mathbf{c})$.

Definition 3.5. A set $X \in \mathbb{R}^n$ is called *open* if $\forall \mathbf{x} \in X$, $\exists \epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset X$.

Definition 3.6. A point $\mathbf{x} \in \mathbb{R}^n$ is a *boundary point* of $X \subset \mathbb{R}^n$ if $\forall \epsilon > 0$, $B_\epsilon(\mathbf{x})$ contains points in X and points not in X. See Figure 3.6.

Definition 3.7. A set $X \in \mathbb{R}^n$ is called *closed* if it contains all of its boundary points.

Example 3.8. Given $\epsilon > 0$, the set

$$D = \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; ||\mathbf{x}|| < \epsilon \text{ and } z \ge 0 \right\}$$

is neither open nor closed in $\mathbb{R}^3$. It contains some but not all of its boundary points.

Definition 3.9. Given $X \in \mathbb{R}^n$, A point $\mathbf{x} \in X$ is an *interior point* of X if $\exists \epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset X$.

We note here that, given $X \subset \mathbb{R}^n$ and an interior point $x \in X$, we call X a *neighborhood* of x. A neighborhood X is open when, of course X is open in $\mathbb{R}^3$. We will often refer to open neighborhoods of a point x without regard to which one we choose.

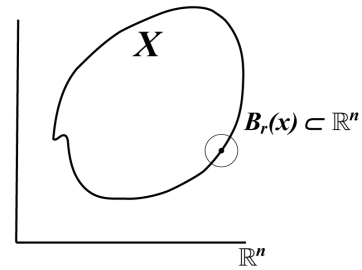

FIGURE 3.6. Any neighborhood of a boundary point $x \in \mathbb{R}^n$ of X will contain points inside X and points outside X.

Here is a better way to "see" a limit without a graph? Separate the domain and codomain spaces. Given a function $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{f}$ has a limit $\mathbf{L}$ at $\mathbf{x} = \mathbf{c}$ if, given any ϵ-ball $B_\epsilon(\mathbf{L})$, one can find a δ-ball $B_\delta(\mathbf{c})$ so the image $\mathbf{f}\left(B_\delta(\mathbf{c})\right)$ lies entirely inside $B_\epsilon(\mathbf{L})$ (except possibly at $\mathbf{c}$).

In practice,

(1) Limits are hard to calculate using the definition, as pathological functions create a diverse array of issues.
(2) Limits follow all of the typical rules found in Calculus I (See page 106.
(3) Most functions in vector calculus are "nice": They behave well on their full domain:
   - vector-valued functions are scalar-valued on each component.
   - Scalar-valued functions involving trig, exponential, logarithmic, rational and polynomial functions are nice even if their arguments involve many variables.

   Example 3.10. The function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \cos(x + y)$ will still have limits everywhere for the same reasons the cosine function did in single variable calculus.

3.2. **Techniques for studying limits.**

3.2.1. *Directional approach.* One technique to study whether a limit exists or not is to reduce the approach of $\mathbf{x}$ to $\mathbf{c}$ to one direction, and use all of the techniques one learns from Calculus I. While this can often be useful for establishing a limit may not exist (coming in from two different direction yields to different values), it is dangerous to use to establish a limit (See accompanying Mathematica files for examples).

Example 3.11. Let $f : \mathbb{R}^2 \to R$ be defined by $f(x, y) = \frac{xy}{x^2+y^2}$. This function is not defined at the origin in $\mathbb{R}^2$. But does the limit exist there? Let's explore by looking only at certain directions. To start, suppose we approach the origin $(0, 0)$, along the linen $y = 0$ in the plane. Then

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,0)\to(0,0)} \frac{x \cdot 0}{x^2 + 0^2} = \lim_{x\to 0} 0 = 0.$$

This result will be the same if we approached the origin along the line $x - 0$ (check this!). But nowm, let's approach the origin in $\mathbb{R}^2$ along the line $y = x$. Here

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,x)\to(0,0)} \frac{x \cdot x}{x^2 + x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

If approaching from different directions yields different values for a limit, then can a limit possibly exist?

Example 3.12. Let $f : \mathbb{R}^2 \to R$ be defined by $g(x, y) = \frac{x^4y^4}{(x^2+y^4)^3}$. This function is again not defined only at the origin in $\mathbb{R}^2$. Does the limit exist at the origin? Let's explore by again looking at certain directions. Suppose we approach the origin $(0, 0)$, along the linen $y = cx$ in the plane. This should help to determine almost every direction of approach depending on the value of $c \in \mathbb{R}$. (which directions are missed?) Then

$$\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{(x,cx)\to(0,0)} \frac{x^4(cx)^4}{(x^2 + (cx)^4)^3} = \lim_{x\to 0} \frac{c^4x^8}{(x^6 + 3c^4x^8 + 3c^8x^{10} + c^{12}x^{12})}$$
$$= \lim_{x\to 0} \frac{c^4x^2}{(1 + 3c^4x^2 + 3c^8x^4 + c^{12}x^6)} = 0.$$

So it would seem here that the limit odes actually exist and is equal to 0 in this case. However, let's approach the origin along the parabola $x = y^2$. Then we have

$$\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{(y^2,y)\to(0,0)} \frac{\left(y^2\right)^4 y^4}{\left((y^2)^2 + y^4\right)^3} = \lim_{y\to 0} \frac{y^{12}}{8y^{12}} = \frac{1}{8}.$$

It turns out that approaching from all directions is more complicated than simply coming in linearly from each direction.

3.2.2. *Polar coordinates.* Switch to polar coordinates and use the fact that $B_\epsilon(\mathbf{x}) = B_\rho(\mathbf{x})$ where ρ is the "distance" variable in the spherical coordinate system on $\mathbb{R}^n$.

Example 3.13. Back to $f : \mathbb{R}^2 \to R$, $f(x, y) = \frac{xy}{x^2+y^2}$, we convert the coordinate system in the plane to polar coordinates through the equations $x = \rho\cos\theta$, and $y = \rho\sin\theta$. Then

$$f(x, y) = f(\rho\cos\theta, \rho\sin\theta) = \frac{(\rho\cos\theta)(\rho\sin\theta)}{(\rho\cos\theta)^2 + (\rho\sin\theta)^2} = \cos\theta\sin\theta = f(\rho, \theta).$$

But approaching from different linear directions to the origin means approaching along lines of fixed θ. As f will take different values for different fixed values of θ, the limit at the origin does not exist.

3.3. **Continuity.** Continuity of functions in vector calculus is pretty much the same as for Calculus I, with a bit of extra structure:

Definition 3.14. A function $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be *continuous at* $\mathbf{a}$ if either $\mathbf{a}$ is an isolated point of X or if

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}).$$

And we say that $\mathbf{f}$ is a continuous function on X if it is continuous at $\mathbf{a}$ for every $\mathbf{a} \in X$.

Some notes:

- In the case of continuity, graphs do not have tears, holes, cliffs, or break in it (this is not a mathematical description).
- Like in Calculus I, sums and scalar multiples of continuous functions are continuous.
- Also, products of continuous functions are continuous, and also quotients where they make sense.

- Compositions of continuous functions are also continuous where they make sense. In this case, we have, if $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : Y \subset \mathbb{R}^m \to \mathbb{R}^p$ are continuous, and $\mathbf{f}(X) \subset Y$, the
$$(\mathbf{g} \circ \mathbf{f}) : X \subset \mathbb{R}^n \to \mathbb{R}^p$$
is continuous.
- the vector-valued function $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{a}$ iff each component function $f_i : X \subset \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$ is continuous at $\mathbf{a}$.