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LECTURE 25: DIFFERENTIAL FORMS.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. A continuation of the last three lectures on differential forms and their structure.

25.1. **More notation.** For $\omega = \sum F_{i_1 i_2 \cdots i_m} dx_{i_1} \wedge \cdots \wedge dx_{i_m}$ a differential m-form on $\mathcal{M} \subset \mathbb{R}^n$, $n \geq m$,

$$\int_{\mathcal{M}} \omega = \underbrace{\int \cdots \int_{\mathcal{M}}}_{n\text{-integrals}} \sum F_{i_1 i_2 \cdots i_m} dx_{i_1} \wedge \cdots \wedge dx_{i_m},$$

where $\mathcal{M}$ is an m-dimensional region in $\mathbb{R}^n$. Note that the order of the form and the dimension of the region integrated over will agree.

Definition 25.1. Let $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1-function. Then the *exterior derivative* of f, denoted df, is the 1-form

$$df = \frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_n} dx_n = Df(\mathbf{x})\, d\mathbf{x} = \nabla f \bullet d\mathbf{x}.$$

For $\omega = \sum F_{i_1 i_2 \cdots i_m} dx_{i_1} \wedge \cdots \wedge dx_{i_m}$ a differential m-form, the differential $(m+1)$-form

$$d\omega = \sum d(F_{i_1 i_2 \cdots i_m}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_m}$$

is called the *exterior derivative* of ω.

Some notes:

- We call a C^1-function $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ a (differential) 0-form. Thus the exterior derivative of a function is simply its differential, a 1-form. Thus theexterior derivative of any differential m-form is always an $(m+1)$-form.
- For each set of indices, the term $d(F_{i_1 i_2 \cdots i_m})$ is the standard differential of a function, and is a 1-form. Upon writing it out, one must then address any and all simplifications and cancellations, which can be many.

Example 25.1. Let $\omega = x^2 y\, dx - x\, dy$ be a C^∞ 1-form on $\mathbb{R}^2$. Then

$$\begin{aligned} d\omega &= d(x^2 y) \wedge dx - d(x) \wedge dy \\ &= (2xy\, dx + x^2\, dy) \wedge dx - (1\, dx - 0\, dy) \wedge dy \\ &= 2xy\, dx \wedge dx + x^2\, dy \wedge dx - dx \wedge dy \\ &= -(1 + x^2)\, dx \wedge dy. \end{aligned}$$

So what is $d(d\omega) = d^2\omega$? (Hint: Is it possible to have a 3-form on the plane?) Here

$$d(d\omega) = d(-(1+x^2))\, dx \wedge dy = -2x\, dx \wedge dx \wedge dy = 0. \quad \text{(Why?)}$$

Example 25.2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as $f(x, y, z) = x^2ye^{2z}$. Calculate df and $d(df) = d^2f$.

First, we have $df = 2xye^{2z}\,dx + x^2e^{2z}\,dy + 2x^2ye^{2z}\,dz$. Then

$$\begin{aligned} d(df) &= d(2xye^{2z}) \wedge dx + d(x^2e^{2z}) \wedge dy + d(2x^2ye^{2z}) \wedge dz \\ &= 2ye^{2z}\,dx \wedge dx + 2xe^{2z}\,dy \wedge dx + 4xye^{2z}\,dz \wedge dx \\ &\quad + 2xe^{2z}\,dx \wedge dy + 0\,dy \wedge dy + 2x^2e^{2z}\,dz \wedge dy \\ &\quad + 4xye^{2z}\,dx \wedge dz + 2x^2e^{2z}\,dy \wedge dz + 4x^2ye^{2z}\,dz \wedge dz \\ &= 0 \end{aligned}$$

due to skew-symmetry and term-by-term cancelations.

But this is a general feature of exterior differentiation, and has broad implications. We say that the exterior derivative is *nilpotent*; it has a positive power, in this case its square, that is 0:

Proposition 25.2. *For ω a differential k-form, $d(d\omega) = d^2\omega = 0$.*

We will not prove this here, but in coordinates, the proof relies on the fact that mixed partials are equal for a sufficiently differentiable function.

Exercise 1. $\omega = F(x, y, z)\,dx + G(x, y, z)\,dy + H(x, y, z)\,dz + J(x, y, z)\,du$, a C^2 1-form on $\mathbb{R}^4$, show that $d^2\omega = 0$.

Here are some other properties of the exterior derivative:

(1) If ω is a k-form, and ν is an ℓ-form, then

$$d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^k \omega \wedge d\nu. \tag{25.1}$$

Note that we call this equation the *Wedge Product Rule* for exterior differentiation.

Exercise 2. Verify, using the Wedge Product Rule, that $d^2(\omega \wedge \nu) = 0$.

(2) As a special case of the Wedge Product Rule, let $k = \ell = 0$. Then $f \wedge g = f \cdot g$, since both f and g are just functions. But then the Wedge Product Rule is simply the Product Rule for the (regular) derivative of functions you learned in Calculus I. Indeed,

$$\begin{aligned} d\left(f(x)g(x)\right) &= d(f \wedge g) = df \wedge g + (-1)^0 f \wedge dg \\ &= f'(x)\,dx \cdot g(x) + f(x) \cdot g'(x)\,dx = \left(f'(x) \cdot g(x) + f(x) \cdot g'(x)\right)\,dx \\ &= df \cdot g + f \cdot dg. \end{aligned}$$

(3) Now look at forms in $\mathbb{R}^3$ only: What one sees is the following:
- There is a one-to-one correspondence between 0-forms and 3-forms:

$$f(x, y, z) \longleftrightarrow f(x, y, z)\,dx \wedge dy \wedge dz.$$

Both have only one term, and the coefficient is just a function on some domain.

- There is a one-to-one correspondence between 1-forms and 2-forms:

$$F_1\,dx + F_2\,dy + F_3\,dz \longleftrightarrow F_1\,dx\wedge\,dy + F_2\,dz\wedge\,dx + F_3\,dy\wedge\,dz.$$

  Both have collections of coefficient functions that can be considered as vector fields on some domain.

(4) Again, only in $\mathbb{R}^3$, look at the effects of the exterior derivative on forms in $\mathbb{R}^3$:
- d(0-form) takes the coefficient function to its gradient.
- d(1-form) takes the coefficient vector field to its curl vector field.
- d(2-form) takes the coefficient vector field to its divergence function.

Perhaps this is another way to think of the ideas that the curl of the gradient is always the zero vector field, and the divergence of the curl of a vector field is always 0. In the language of differential forms on $\mathbb{R}^3$, they both are just $d^2\omega = 0$.

Theorem 25.3 (Generalized Stokes' Theorem)**.** *Let $\mathcal{D}\subset\mathbb{R}^k$ be a compact region with nonempty interior, and $\mathcal{M} = \mathbf{X}(\mathcal{D})$ be an oriented, parameterized k-dimensional hypersurface in $\mathbb{R}^n$, with $k\le n$ and $\partial\mathcal{M}$ oriented compatibly. Then, for a $(k-1)$-form defined on an open set in $\mathbb{R}^n$ containing $\mathcal{M}$, we have*

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega.$$

Note: if $\mathcal{M}$ is closed, so that $\partial\mathcal{M} = \emptyset$, then $\int_{\partial\mathcal{M}}\omega = 0$, since integrating over nothing is nothing.

Now we are in a position to understand the big theorems that we have already studied individually.

25.2. **In the language of forms, The Theorem of Gauss.** In Theorem 25.3, let $n = k = 3$, and $\mathcal{M}$ be a (3-dimensional) compact solid in $\mathbb{R}^3$, with

$$\omega = F_1(\mathbf{x})\,dy\wedge\,dz + F_2(\mathbf{x})\,dz\wedge\,dx + F_3(\mathbf{x})\,dx\wedge\,dy$$

a differential 2-form on a superset of $\mathcal{M}$ in $\mathbb{R}^3$. Then, we know that ω is a 2-form on the closed surface $\partial\mathcal{M}$, and that one interpretation of the integral of ω over $\partial\mathcal{M}$ is just the vector surface integral of the vector field $\mathbf{F}(\mathbf{x}) = F_1(\mathbf{x})\,\mathbf{i} + F_2(\mathbf{x})\,\mathbf{j} + F_3(\mathbf{x})\,\mathbf{k}$ over the surface, so

$$\int_{\partial\mathcal{M}} \omega = \oiint_{\partial\mathcal{M}} \mathbf{F}\bullet d\mathbf{S}, \quad \text{where} \quad d\mathbf{S} = \begin{bmatrix} dy\wedge\,dz \\ dz\wedge\,dx \\ dx\wedge\,dy \end{bmatrix}.$$

This is the left-hand-side of Gauss' Theorem.

For the right-hand-side of Gauss' Theorem, note that $d\omega$ will be a 3-form. We have

$$\begin{aligned}
\int_{\mathcal{M}} d\omega &= \int_{\mathcal{M}} d\left(F_1(\mathbf{x})\, dy \wedge\, dz + F_2(\mathbf{x})\, dz \wedge\, dx + F_3(\mathbf{x})\, dx \wedge\, dy\right) \\
&= \int_{\mathcal{M}} \left(\frac{\partial F_1}{\partial x}\, dx + \frac{\partial F_1}{\partial y}\, dy + \frac{\partial F_1}{\partial z}\, dz\right) \wedge\, dy \wedge\, dz \\
&\qquad + \left(\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz\right) \wedge\, dz \wedge\, dx \\
&\qquad + \left(\frac{\partial F_3}{\partial x}\, dx + \frac{\partial F_3}{\partial y}\, dy + \frac{\partial F_3}{\partial z}\, dz\right) \wedge\, dx \wedge\, dy \\
&= \int_{\mathcal{M}} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge\, dy \wedge\, dz = \int_{\mathcal{M}} (\mathbf{div}\, \mathbf{F})\, dV.
\end{aligned}$$

Note that all summands with like terms in their wedge products are 0, and every permutation needed to make the only surviving term $dx \wedge dy \wedge dz$ introduces a minus sign, but there are an even number of permutations to generate the coefficient sum. The end result is precisely the right-hand-side of Gauss' Theorem. So, when the dimensions match, *The Generalized Stokes' Theorem is Gauss' theorem.*

25.3. **In the language of forms, The Theorem of Stokes.** In Theorem 25.3, let $k = 2$ and $n = 3$. In this case, let $\mathcal{D} \subset \mathbb{R}^2$ be a compact region (with boundary), and $\mathcal{S} = \mathbf{X}(\mathcal{D}) \subset \mathbb{R}^3$ be an oriented parameterized surface, with the closed curve $\partial\mathcal{S}$ oriented compatibly. And let

$$\omega = F_1(\mathbf{x})\, dx + F_2(\mathbf{x})\, dy + F(\mathbf{x})\, dz$$

be a differential 1-form, defined on a superset of $\mathcal{S}$ in $\mathbb{R}^3$. Then, we know that ω is a 1-form on the closed curve $\partial\mathcal{S}$, and that one interpretation of the integral of ω over $\partial\mathcal{S}$ is just the vector line integral (the circulation) of the vector field $\mathbf{F}(\mathbf{x}) = F_1(\mathbf{x})\,\mathbf{i} + F_2(\mathbf{x})\,\mathbf{j} + F_3(\mathbf{x})\,\mathbf{k}$ over the curve. So

$$\int_{\partial\mathcal{S}} \omega = \oint_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s}, \quad \text{where} \quad d\mathbf{s} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.$$

This is the left-hand-side of Stokes' Theorem.

For the right-hand-side of Stokes' Theorem, note that $d\omega$ will be a 2-form. We have

$$\begin{aligned}
\int_{\mathcal{S}} d\omega &= \int_{\mathcal{S}} d\left(F_1(\mathbf{x})\,dx + F_2(\mathbf{x})\,dy + F_3(\mathbf{x})\,dz\right) \\
&= \int_{\mathcal{S}} \left(\frac{\partial F_1}{\partial x}\,dx + \frac{\partial F_1}{\partial y}\,dy + \frac{\partial F_1}{\partial z}\,dz\right) \wedge dx \\
&\qquad + \left(\frac{\partial F_2}{\partial x}\,dx + \frac{\partial F_2}{\partial y}\,dy + \frac{\partial F_2}{\partial z}\,dz\right) \wedge dy \\
&\qquad + \left(\frac{\partial F_3}{\partial x}\,dx + \frac{\partial F_3}{\partial y}\,dy + \frac{\partial F_3}{\partial z}\,dz\right) \wedge dz \\
&= \int_{\mathcal{M}} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial z}\right) dx \wedge dy \\
&= \int_{\mathcal{S}} \nabla \times \mathbf{F} \bullet d\mathbf{S}.
\end{aligned}$$

As before, all summands with like terms in their wedge products are 0, and the minus signs come from the permutations needed to combine the remaining terms, if possible. The end result is precisely the right-hand-side of Stokes' Theorem. So, again, when the dimensions are right, *The Generalized Stokes' Theorem is Stokes' Theorem.*

25.4. **In the language of forms, The Theorem of Green.** Once more in Theorem 25.3, let $k = n = 2$. In this case, let $\mathcal{D} \subset \mathbb{R}^2$ be a compact region (with boundary), and

$$\omega = F_1\,dx + F_2(\mathbf{x})\,dy$$

be a differential 1-form, defined on a superset of $\mathcal{D}$ in $\mathbb{R}^2$. As in the discussion above involving Stokes' Theorem, integrating ω over $\partial\mathcal{D}$ is akin to calculating the circulation of $\mathbf{F}$ over $\partial\mathcal{D}$, so

$$\int_{\partial\mathcal{D}} \omega = \oint_{\partial\mathcal{D}} \mathbf{F} \bullet d\mathbf{s} = \oint_{\partial\mathcal{D}} F_1\,dx + F_2\,dy = \oint_{\partial\mathcal{D}} M(x,y)\,dx + N(x,y)\,dy,$$

where here we expose the notation used in Green's Theorem by setting $M(x,y) = F_1(x,y)$ and $N(x,y) = F_2(x,y)$. This is the left-hand-side of Green's Theorem.

For the right-hand-side of Green's Theorem, note that $d\omega$ will be a 2-form. We have

$$\begin{aligned}
\int_{\mathcal{D}} d\omega &= \int_{\mathcal{D}} d\left(M(x,y)\,dx + N(x,y)\,dy\right) = \int_{\mathcal{D}} dM \wedge dx + dN \wedge dy \\
&= \int_{\mathcal{D}} \left(\frac{\partial M}{\partial x}\,dx + \frac{\partial M}{\partial y}\,dy\right) \wedge dx + \left(\frac{\partial N}{\partial x}\,dx + \frac{\partial N}{\partial y}\,dy\right) \wedge dy \\
&= \int_{\mathcal{D}} \frac{\partial M}{\partial x}\,dx \wedge dx + \frac{\partial M}{\partial y}\,dy \wedge dx + \frac{\partial N}{\partial x}\,dx \wedge dy + \frac{\partial N}{\partial y}\,dy \wedge dy \\
&= \int_{\mathcal{D}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \wedge dy = \iint_{\mathcal{D}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.
\end{aligned}$$

Again, the end result is precisely the right-hand-side of Green's Theorem. And, once again, when the dimensions are right, *The Generalized Stokes' Theorem is Green's Theorem.*

25.5. **In the language of forms, The Fundamental Theorem of Calculus.** One last time, suppose that in Theorem 25.3, we let $k = n = 1$. Then, for $\mathcal{I} = [a, b] \subset \mathbb{R}$ a closed, bounded, (compact) interval in $\mathbb{R}$, with $\partial\mathcal{I} = \{a, b\}$ 2 points, and for ω a 0-form (just a function) on a superset of $\mathcal{I}$ in $\mathbb{R}$. Then, upon orienting $\mathcal{I}$, we automatically orient $\partial\mathcal{I}$. Going from a to b renders the orientation on $\partial\mathcal{I}$ in such a fashion that the upper endpoint is considered positive and the lower endpoint is considered negative. Using this,

$$\begin{aligned}\int_{\partial\mathcal{I}} \omega &= \text{adding up all values of } f(x) \text{ on the set of} \\ &\qquad \text{points } \{a, b\}\,, \text{ oriented compatibly with } \mathcal{I} \\ &= f(b) - f(a).\end{aligned}$$

This is just the right-hand-side of the Fundamental Theorem of Calculus.

Note: We have yet had no reason to understand the orientation of a discrete set of points, or a 0-dimensional set. One does this simply by assigning a plus or minus to each point separately. By convention, then, the orientation induced on the boundary of an interval upon orienting the interval assigns a minus sign to the lower point, and a plus to the higher point. So here, $f(a)$ is considered negative, and $f(b)$ is considered positive.

And for ω a 0-form (a function), We know that $d\omega$ will be a 1-form (its differential). We have

$$\int_{\mathcal{I}} d\omega = \int_{\mathcal{I}} df = \int_a^b f'(x)\, dx.$$

But putting these together, we get

$$\int_{\partial\mathcal{I}} \omega = \underbrace{f(b) - f(a) = \int_a^b f'(x)\, dx}_{\text{Fund. Thm of Calc.}} = \int_{\mathcal{I}} d\omega.$$

Hence, *The Generalized Stokes' Theorem is also just the Fundamental Theorem of Calculus.*

Put all of this together and one can easily see that Theorem 25.3, being a dimensionless, and coordinate-less statement on a relationship between quantities defined on a region and related quantities restricted to its boundary, is just *The Fundamental Theorem of Vector Calculus.*