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LECTURE 23: DIFFERENTIAL FORMS.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. During these last three lectures, I will discuss the structure of differential forms form the perspective of multi-linear algebra and n-forms on vector spaces. This is basically not done in the book. This allows me to give a much more foundational treatment of just what forms are and not just how they work. We learn their structure, how to integrate them and how to differentiate them, all with an eye toward what works regardless of the dimension. We show how many of the things we learned in the past, from the product rule and the Substitution Method in Calculus I to the Change of Variables Theorem and Fubini's Theorem in Calculus III, are all just examples of more general structure. We then finish with the Generalized Stoke's Theorem, and show how the various big theorems of Gauss, Stokes and Green are also simply particular examples. We end with the same result of the Fundamental Theorem of Calculus. In fact, one can easily say that the Generalized Stokes Theorem is just the Fundamental Theorem of Multivariable Calculus.

23.1. **Multilinear algebra.** Let V be an n-dimensional vector space on $\mathbb{R}$. Then, relative to some basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, (note that we will use the notation of the *standard basis* in $\mathbb{R}^n$ for convenience, but one can adapt this argument to any basis) any element $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = v_1\mathbf{e}_1 + \ldots + v_n\mathbf{e}_n.$$

Here, v_i is the ith coordinate of $\mathbf{v}$ (in the given basis) and, by convention, one often denotes elements of V by their set of coordinates in the form of a *(column) vector*

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in V, \quad v_i \in \mathbb{R}, \quad \text{for } i = 1, \ldots, n,$$

so that we say $\mathbf{v} \in V$ is a *vector* in V.

A *linear functional*, or *(linear) 1-form*, or *covector*, is a linear map $f : V \to \mathbb{R}$, so satisfies

$$f(c_1\mathbf{v} + c_1\mathbf{w}) = c_1 f(\mathbf{v}) + c_1 f(\mathbf{w}), \quad \forall \mathbf{v}, \mathbf{w} \in V, \quad \forall c_1, c_2 \in \mathbb{R}.$$

The set of all covectors of V is again a n-dimensional vector space V^* called the *dual space* to V.

Exercise 1. Show that V^* is a vector space.

So what is a basis for V^*? For each $i = 1, \ldots, n$, let $\mathbf{e}_i : V \to \mathbb{R}$ be defined so that

$$\mathbf{e}_i^*(\mathbf{e}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Then, by linearity, $\mathbf{e}_i^*(\mathbf{w}) = w_i$, and the ith basis covector strips off the ith entry of $\mathbf{w}$. One can readily show that $\{\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*\}$ forms a basis for V^*, and that for any $\mathbf{v}^* \in V^*$,

$$\mathbf{v}^* = v_1\mathbf{e}_1^* + \ldots + v_n\mathbf{e}_n^*, \quad v_i \in \mathbb{R}.$$

Here, $\mathbf{v}^* : V \to \mathbb{R}$ satisfies

$$\begin{aligned}
\mathbf{v}^*(\mathbf{w}) &= v_1\mathbf{e}_1^*(\mathbf{w}) + \ldots + v_n\mathbf{e}_n^*(\mathbf{w}) \\
&= v_1 w_1 + \ldots + v_n w_n = \mathbf{v} \bullet \mathbf{w} \\
&= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}.
\end{aligned}$$

Some notes:

- In this way, we often write covectors as *row vectors*, since written this way, they can readily "act" on vectors as functionals.
- The dot product $\mathbf{dot} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is not a linear function. It is linear, however, on each of its factors separately, and is an example of a *mulitlinear function*. Indeed, for $v \in V$, the dot product $\mathbf{dot}(\mathbf{v}, \cdot) = \mathbf{dot}_{\mathbf{v}}(\cdot)$ with one slot filled is a linear functional, so that one can identify, for $\mathbf{v}^* \in V^*$,

  $$\mathbf{v}^*(\mathbf{w}) = \mathbf{dot}_{\mathbf{v}}(\mathbf{w}) = \mathbf{v} \bullet \mathbf{w},$$

  as before.
- In $\mathbb{R}^3$, each $\mathbf{p} \in \mathbb{R}^3$ has a *tangent space* $T_{\mathbf{p}}\mathbb{R}^3$, which is another copy of $\mathbb{R}^3$, but with its origin based at $\mathbf{p}$. It is a different space than the one where $\mathbf{p}$ "lives".

On this last bullet point, for coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}^n$, define a coordinate system on $T_{\mathbf{p}}\mathbb{R}^n$ as $(dx_1, \ldots, dx_n)$, where each dx_i is the infinitesimal change in the coordinate x_i at $\mathbf{p}$ in $\mathbb{R}^n$, but ranges over all real numbers in a particular direction in $T_{\mathbf{p}}\mathbb{R}^n$. Here, each dx_i is a linear functional on $T_{\mathbf{p}}\mathbb{R}^n$ since, for a choice of $\mathbf{v} \in T_{\mathbf{p}}\mathbb{R}^n$, $dx_i(\mathbf{v}) = v_i$.

Some notes:

- Think of a parameterized hypersurface $\mathcal{S} \in \mathbb{R}^n$, and it is easier to see how a tangent vector $\mathbf{v} \in T_{\mathbf{p}}\mathcal{S}$, but $\mathbf{v} \notin \mathcal{S}$.
- This definition of dx_i works because coordinates themselves are actually linear functionals on a space (at least the Cartesian one are). They are projections onto the factors of the space, which are linear functions. INdeed, let $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2$. Then the functions $x : \mathbb{R}^2 \to \mathbb{R}$ and $y : \mathbb{R}^2 \to \mathbb{R}$ can be defined as $x(\mathbf{p}) = p_1$ and $y(\mathbf{p}) = p_2$. These coordinate functions are linear and hence are not only continuous but differentiable, and the derivative functions are

  $$\begin{aligned}
  Dx_{\mathbf{p}} &: T_{\mathbf{p}}\mathbb{R}^2 \to \mathbb{R}, \quad Dx_{\mathbf{p}} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \text{and} \\
  Dy_{\mathbf{p}} &: T_{\mathbf{p}}\mathbb{R}^2 \to \mathbb{R}, \quad Dy_{\mathbf{p}} = \begin{bmatrix} 0 & 1 \end{bmatrix},
  \end{aligned}$$

where each is a 1×2-matrix. Then, given $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in T_{\mathbf{p}}\mathbb{R}^2$, we have

$$Dx_{\mathbf{p}}(\mathbf{v}) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1,$$
$$Dx_{\mathbf{p}}(\mathbf{v}) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1.$$

So we use this to <u>define</u> coordinates directly inside $T_{\mathbf{p}}\mathbb{R}^2$, (dx, dy), where

$$dx = Dx_{\mathbf{p}} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad dy = Dy_{\mathbf{p}} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Example 23.1. Let $\mathbf{v} \in \mathbb{R}^3$, so $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$. Then we can write

$$\begin{array}{ll} x : \mathbb{R}^3 \to \mathbb{R} & x(\mathbf{v}) = \mathbf{v} \bullet \mathbf{i} = \mathbf{e}_1^*(\mathbf{v}) = v_1, \\ y : \mathbb{R}^3 \to \mathbb{R} & y(\mathbf{v}) = \mathbf{v} \bullet \mathbf{j} = \mathbf{e}_2^*(\mathbf{v}) = v_2, \\ z : \mathbb{R}^3 \to \mathbb{R} & z(\mathbf{v}) = \mathbf{v} \bullet \mathbf{k} = \mathbf{e}_3^*(\mathbf{v}) = v_3. \end{array}$$

Defined, implicitly at least, this way, we often "abuse notation" for convenience and clarity of concept and simply write

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3,$$

as one would normally see in a calculus text.

Geometrically, a linear functional on $\mathbb{R}^n$ looks like

$$\omega = a_1\,dx_1 + \ldots + a_n\,dx_n = \mathbf{a}\,d\mathbf{x},$$

where the (row matrix) covector $\mathbf{a}$ is called the *coefficient vector* of the functional, and $d\mathbf{x} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$ is a basis of covectors (linear functionals) in $\mathbb{R}^n$. However, we could write $\mathbf{a}$ as a column vector. If we did, then, we would be forced to write $\omega = \mathbf{a} \bullet d\mathbf{x}$. We do see this at times, and context should make it clear. See Equation 23.1 below.

Example 23.2. Let $\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix} \in \mathbb{R}^3$. Then

$$\omega = \mathbf{a}\,d\mathbf{x} = a_1\,dx + a_2\,dy + a_3\,dz = dx + 2\,dy + 3\,dz,$$

and

$$\begin{aligned} \omega(\mathbf{v}) &= a_1\,dx(\mathbf{v}) + a_2\,dy(\mathbf{v}) + a_3\,dz(\mathbf{v}) \\ &= a_1v_1 + a_2v_2 + a_3v_3 = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix} \\ &= 1(-4) + 2(-5) = 3(-6) = -32. \end{aligned}$$

Example 23.3. It is also a good idea to keep in mind where different mathematical objects "live": Let $\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \in T_{\mathbf{p}}\mathbb{R}^2$, for $\mathbf{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then, while we envision $\mathbf{v}$ as a vector in $\mathbb{R}^2$ based at $\mathbf{p}$, it is really a vector based at the origin (like vectors should) of $T_{\mathbf{p}}\mathbb{R}^2$, considered a different space.

Let $\mathbf{c} : [a, b] \to \mathbb{R}^2$ be a C^1-curve. For $t_0 \in (a, b)$, $\mathbf{p} = \mathbf{c}(t_0) \in \mathbb{R}^2$, the space $T_{\mathbf{p}}\mathbb{R}^2$ is not the same plane as $\mathbb{R}^2$. For one, it has different coordinates. We can write the tangent line $\ell_{\mathbf{p}}$ to the curve at $\mathbf{p}$ via the coordinates (dx, dy) of $T_{\mathbf{p}}\mathbb{R}^2$; This is because $\ell_{\mathbf{p}}$ is defined as the set of all tangent vectors to $\mathbf{c}$ at $\mathbf{p}$, so $\ell_{\mathbf{p}} \subset T_{\mathbf{p}}\mathbb{R}^2$, and not really in the plane where $\mathbf{c}$ is defined. In fact, the line $\ell_{\mathbf{p}}$ is a *vector subspace* of $T_{\mathbf{p}}\mathbb{R}^2$: The equation for $\ell_{\mathbf{p}}$ is

$$dy = (\text{constant})\, dx.$$

Can you guess what the constant is?

Example 23.4. Let $\mathbf{c} : [0, 4] \to \mathbb{R}^2$ be defined by $\mathbf{c}(t) = (t, t^2)$. In the xy-plane, the equation for the tangent line to $\mathbf{c}$ at $\mathbf{p} = \mathbf{c}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is

$$(y - 1) = 2(x - 1), \quad \text{or} \quad y = 2x - 1.$$

However, in $T_{\mathbf{p}}\mathbb{R}^2$, a copy of $\mathbb{R}^2$, but with the origin at $\mathbf{p}$ and coordinates (dx, dy), the equation for $\ell_{\mathbf{p}}$ is

$$dy = 2\, dx, \quad \text{or} \quad \frac{dy}{dx} = 2.$$

Just for contrast, the equation for $\ell_{\mathbf{q}} \in T_{\mathbf{q}}\mathbb{R}^2$, when $\mathbf{q} = \mathbf{c}(3) = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ is $dy = 6\, dx$.

Now compare this to the de-parameterized curve: Let $x = t$, so that $y = f(x) = x^2$. Now the curve $\mathbf{c}$ is the graph of the function $f : [0, 4] \to \mathbb{R}$ (and parameterized by x). Using dy as the infinitesimal change in y, its relationship to dx, an infinitesimal change in x, is then $dy = f'(x)\, dx = 2x\, dx$. And we are back in Calculus I.

And now, we can generalize:

Definition 23.5. A *one form* on a smooth region $\mathcal{D} \subset \mathbb{R}^n$ is a choice of a linear one form on each tangent space to $\mathcal{D}$ which varies continuously with $\mathbf{p} \in \mathcal{D}$.

Some notes:

- This definition sounds a lot like that of a vector field, a choice of a vector in each tangent space to $\mathcal{D}$ which varies continuously with $\mathbf{p} \in \mathcal{D}$. It is actually quite close!
- Instead of a vector choice in our vector field, a one-form is a choice of a covector, or linear functional, in each tangent space. In this sense, a 1-form on $\mathcal{D}$ is a *covector field* on $\mathcal{D}$.

On $\mathbb{R}$, a generic 1-form looks like $\omega = f(x)\, dx$, for f a C^0-function on $\mathbb{R}$. At a point $x_0 \in \mathbb{R}$, $f(x_0) = a$, and the linear functional (the covector) at $T_{x_0}\mathbb{R}$, which is a copy of $\mathbb{R}$ but with the origin at x_0, is $\omega_{x_0} = a\, dx$. Then, for $v \in T_{x_0}\mathbb{R}$, we have

$$\omega_{x_0}(v) = a\, dx(v).$$

On $\mathbb{R}^n$, a generic 1-form looks like

$$\omega = f_1(\mathbf{x})\,dx_1 + \ldots + f_n(\mathbf{x})\,dx_n = \sum_{i=1}^{n} f_i(\mathbf{x})\,dx_i = \mathbf{F} \bullet d\mathbf{x}, \tag{23.1}$$

where $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}$, and $d\mathbf{x} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$. Do you recognize this formula? A common way to construct 1-forms on a domain is to use vector fields as the *coefficient functions* of the form. But really, a 1-form is a covector field. We are simply writing the coefficients a column vectors instead of their more properly written row vectors. But this is what we alluded to in the discussion just after Example 23.1

Some final notes:

- This $d\mathbf{x}$ is precisely the $d\mathbf{s}$ in the definition of the vector line integral $\int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s}$. In a sense, integrating a vector field along a curve IS just the adding up of the values of a 1-form along the curve.
- A 1-form is called a *differential 1-form* if the coefficient functions $f_i(\mathbf{x})$ are C^1-functions for all $i = 1, \ldots, n$.
- For any real-valued C^2-function $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$, its *differential*

$$df(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x})\,dx_i = \frac{\partial f}{\partial x_1}(\mathbf{x})\,dx_1 + \ldots + \frac{\partial f}{\partial x_n}(\mathbf{x})\,dx_n$$

  is a differential 1-form since each of the functions $\frac{\partial f}{\partial x_i}(\mathbf{x})$ is a C^1-function. But 1-forms do not have to arise in this fashion (that is, not all 1-forms arise as the differentials of functions).