April 17, 2019

LECTURE 21: THE THEOREM OF STOKES'.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. In this lecture, we begin to finish the foundational material of what makes a vector calculus course with a full discussion of one of the two other Big Theorems, those of Stokes and Gauss. Here, we present and discuss Stokes' Theorem, developing the intuition of what the theorem actually says, and establishing some main situations where the theorem is relevant. Then we use Stokes' Theorem in a few examples and situations.

Theorem 21.1 (Stokes' Theorem)**.** *Let $\mathcal{S}$ be a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$, where $\partial\mathcal{S}$ consists of finitely many piecewise smooth closed curves oriented compatibly. FOr $\mathbf{F}$ a C^1-vector field on a domain containing $\mathcal{S}$,*

$$\iint_{\mathcal{S}} \nabla \times \mathbf{F} \bullet d\mathbf{S} = \oint_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s}.$$

Some notes:

(1) Here, the surface integral of the curl of a vector field along a surface is equal to the circulation of the vector field along the boundary of the surface.
(2) This is a lot like Green's Theorem:
   - The left-hand side measure the normal component of the curl of $\mathbf{F}$ along $\mathcal{S}$, so measures the amount of twisting in the direction through $\mathcal{S}$).
   - The right-hand side measures the tangent component of $\mathbf{F}$ along $\partial\mathcal{S}$.
(3) In a way, the shape of the surface doesn;t matter as much as what is happening on the boundary. According to Stokes' Theorem, in each of the surfaces in Figure **??**, the value of $\iint_{\mathcal{S}} \nabla \times \mathbf{F} \bullet d\mathbf{S}$ is the same.
(4) Typical use: Sometimes the flux of the curl of $\mathbf{F}$ is hard to calculate across a bounded surface. But the circulation along its boundary is not!

Example 21.2. ***Compute the flux of the curl of*** $\mathbf{F} = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ ***through the surface of the sphere*** $x^2+y^2+z^2=4$ ***inside the cylinder*** $x^2+y^2=1$ ***and above the*** xy***-plane in*** $\mathbb{R}^3$***.*** The strategy for this calculation is that, since both the vector field and the surface satisfy Stokes' Theorem ($\mathbf{F}$ is C^1 and $\mathcal{S}$, using the outward normal is orientable and bounded with a closed, smooth boundary curve, which we can orient compatibly as counterclockwise, or with $\mathcal{S}$ on the left, walking upright on the curve), we look to calulate the surface integral by instead calculating the circulation of $\mathbf{F}$ along $\partial\mathcal{S}$.

First, let's parameterize $\partial\mathcal{S}$ so that the orientation given by the parameterization is compatible with the surface orientation. Here, $\partial\mathcal{S}$ is on both the sphere $x^2+y^2+z^2=4$, as well as the cylinder $x^2+y^2=1$. Hence

$$(x^2+y^2)+z^2 = 1+z^2 = 4,$$

so that $z^2 = 3$ and $z = \sqrt{3}$ (recall we are only using the positive hemisphere here). So parameterize $\partial\mathcal{S}$ as $\mathbf{c} : [0, 2\pi] \to \mathbb{R}^3$ via

$$\mathbf{c}(t) = \begin{bmatrix} \cos t \\ \sin t \\ \sqrt{3} \end{bmatrix} \in \mathbb{R}^3.$$

The next step is to calculate the circulation of $\mathbf{F}$ over $\mathbf{c}$. Here we have

$$\begin{aligned} \iint_{\mathcal{S}} \nabla \times \mathbf{F} \bullet d\mathbf{S} \overset{\textbf{Stokes'}}{=\!=\!=} \int_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s} &= \int_{\mathbf{c}} \mathbf{F}\left(\mathbf{c}(t)\right) \bullet \mathbf{c}'(t)\, dt \\ &= \int_0^{2\pi} \begin{bmatrix} \sqrt{3}\cos t \\ \sqrt{3}\sin t \\ \cos t \sin t \end{bmatrix} \bullet \begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix} dt \\ &= \int_0^{2\pi} 0\, dt = 0. \end{aligned}$$

Now suppose that we did this calculation directly, without using Stoke's. In this case, the strategy is to parameterize the surface, which we will do using spherical coordinates (note that the "curved disk" here, which comprises $\mathcal{S}$, lies on the $\rho = 4$ sphere). In spherical coordinates, we find that the region is, in fact, a rectangle in the two angles. Then we calculate the resulting double integral.

Here, we start with calculating $\mathbf{curl}(\mathbf{F})$:

$$\mathbf{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} = (x-y)\,\mathbf{i} - (y-x)\,\mathbf{j} + 0\,\mathbf{k} = \begin{bmatrix} x-y \\ x-y \\ 0 \end{bmatrix}.$$

Next, we parameterize $\mathcal{S}$. First, we again note that the entirety of $\mathcal{S}$ lies on the sphere at $\rho = 2$. This leaves us already with the parameterization $\mathbf{X} : \mathcal{D} \to \mathbb{R}^3$ based solely on the angles

$$\mathbf{X}(\theta\varphi) = \begin{bmatrix} 2\cos\theta\sin\varphi \\ 2\sin\theta\sin\varphi \\ 2\cos\varphi \end{bmatrix}.$$

Second, we seek to define the region $\mathcal{D}$ in the $\theta\varphi$-plane that leads to $\mathcal{S} = \mathbf{X}(\mathcal{D})$. To this end, we note that the azimuth angle goes all the way to describe $\mathcal{S}$, so $\theta \in [0, 2\pi]$. However, the polar angle φ will only go from $\varphi = 0$, at the north pole, to the edge of $\mathcal{S}$, so we need to find the value of φ that corresponds to the edge. To see this, look directly into the xz-plane, and note that the $\rho = 2$ sphere forms a semicircle of radius 2, and the intersection of $\partial\mathcal{S}$ in this plane occurs at a point on this semicircle with x-coordinate 1. One can calculate that the y-coordinate here is $\sqrt{3}$, and that the radial line from the origin to $\partial\mathcal{S}$ has angle $\alpha = \frac{\pi}{3}$. This means that the polar angle $\varphi = \frac{\pi}{2} - \alpha = \frac{\pi}{6}$.

Hence the region $\mathcal{D}$ in the $\theta\varphi$-plane corresponds to $\mathcal{D} = [0, 2\pi] \times \left[0, \frac{\pi}{6}\right]$.

Next, before we integrate, we need to check to ensure that our idea of orienting $\mathcal{S}$ with the normal pointing outward is correct, using this parameterization. We need this since we have already oriented our curve $\mathbf{c}$ in the previous calculation to be compatible with the outward pointing normal. Here, we have

$$\mathbf{X}_\theta = \begin{bmatrix} -2\sin\theta\sin\varphi \\ 2\cos\theta\sin\varphi \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{X}_\varphi = \begin{bmatrix} 2\cos\theta\cos\varphi \\ 2\sin\theta\cos\varphi \\ -2\sin\varphi \end{bmatrix},$$

and $\mathbf{N} = \mathbf{X}_\theta \times \mathbf{X}_\varphi = \begin{bmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ \cos\varphi \end{bmatrix}(-4\sin\varphi)$. But this means

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{X}_\theta \times \mathbf{X}_\varphi||} = -\begin{bmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ \cos\varphi \end{bmatrix}.$$

Unfortunately, this unit normal points inward (toward the origin). This is fine, but it is incompatible with our orientation of $\partial\mathcal{S}$. To fix this, we simply reparameterize to generate the other orientation. The simplest way to do this is to rewrite the region $\mathcal{D}$ as lying inside the $\varphi\theta$-place instead of the $\theta\varphi$-plane. Then the region $\mathcal{D} = \left[0, \frac{\pi}{6}\right] \times [0, 2\pi]$ in the $\varphi\theta$-plane and the normal vector to $\mathcal{S}$ using this new orientation reversing reparameterization will be

$$\mathbf{N} = \mathbf{X}_\varphi \times \mathbf{X}_\theta = -\mathbf{X}_\theta \times \mathbf{X}_\varphi$$

by the properties of the cross product.

Lastly, we calculate the flux of the curl:

$$\begin{aligned}
\iint_{\mathcal{S}} \nabla \times \mathbf{F} \bullet d\mathbf{S} &= \iint_{\mathcal{D}} \nabla \times \mathbf{F}\left(\mathbf{X}(\varphi,\theta)\right) \bullet \left(\mathbf{X}_\varphi \times \mathbf{X}_\theta\right)\, dA \\
&= \iint_{\mathcal{D}} \left(\nabla \times \mathbf{F}\left(\mathbf{X}(\varphi,\theta)\right) \bullet \mathbf{n}\right)\, dA \\
&= \int_0^{\frac{\pi}{6}} \int_0^{2\pi} \begin{bmatrix} 2\sin\varphi(\cos\theta - \sin\theta) \\ 2\sin\varphi(\cos\theta - \sin\theta) \\ 0 \end{bmatrix} \bullet \begin{bmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ \cos\varphi \end{bmatrix} (4\sin\varphi)\, d\theta\, d\varphi \\
&= \int_0^{\frac{\pi}{6}} \int_0^{2\pi} 8\sin^3\varphi \left(\cos^2\theta - \sin^2\theta\right)\, d\theta\, d\varphi \\
&= \int_0^{\frac{\pi}{6}} 8\sin^3\varphi \left(\int_0^{2\pi} \sin 2\theta\, d\theta\right)\, d\varphi.
\end{aligned}$$

But the inside integral is 0, since

$$\int_0^{2\pi} \sin 2\theta\, d\theta = \left[-\frac{1}{2}\cos 2\theta\Big|_0^{2\pi}\right] = 0.$$

Hence the entire double integral is 0.

So which was easier??

Here is a less typical example of the use of Stokes' Theorem: Sometimes, one can use Stokes' to change the surface in a way that leaves the boundary fixed. So if a calculation of the flux of the curl of a vector field across $\mathcal{S}$ is difficult, and the circulation of the vector field along $\partial\mathcal{S}$ is also difficult, if Stokes applies, one can just find a different surface, with the same boundary, where the flux of the curl is easier to integrate.

Example 21.3. ***Example 7.3.2 of the text.*** Calculate the flux of the curl of

$$\mathbf{F} = \begin{bmatrix} e^{y+z} - 2y \\ xe^{y+z} + y \\ e^{x+y} \end{bmatrix} \quad \text{across} \quad \mathcal{S} = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, z \geq \frac{1}{e},\ z = e^{-(x^2+y^2)} \right\}.$$

By inspection and a few quick calculations (check the book), one can use Stokes' Theorem here, but both sides of the equal sign in the theorem are quite difficult calculations! However, by Stokes', <u>any</u> surface with the same boundary as $\mathcal{S}$ will do, when calculating the flux of the curl of $\mathbf{F}$ across it.

So here, choose

$$\widehat{\mathcal{S}} = \left\{ \left(x, y, \frac{1}{e}\right) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 \leq 1 \right\}.$$

Then, we have

$$\partial\widehat{\mathcal{S}} = \partial\mathcal{S} = \left\{ \left(x, y, \frac{1}{e}\right) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 = 1 \right\}.$$

So by Stokes' Theorem,

$$\iint_{\mathcal{S}} \nabla \times \mathbf{F} \bullet d\mathbf{S} = \iint_{\widehat{\mathcal{S}}} \nabla \times \mathbf{F} \bullet d\mathbf{S} = \iint_{\widehat{\mathcal{S}}} (\nabla \times \mathbf{F} \bullet \mathbf{n})\, dS.$$

So for this calculation, we find

$$\nabla \times \mathbf{F} = \begin{bmatrix} e^{x+y} - xe^{y+z} \\ e^{y+z} - e^{x+y} \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So then $\nabla \times \mathbf{F} \bullet \mathbf{n} = 2$. Now the calculation is simply

$$\begin{aligned} \iint_{\mathcal{S}} \nabla \times \mathbf{F} \bullet d\mathbf{S} &= \iint_{\widehat{\mathcal{S}}} (\nabla \times \mathbf{F} \bullet \mathbf{n})\, dS \\ &= \iint_{\widehat{\mathcal{S}}} 2\, dS = 2\left(\mathbf{area}(\widehat{\mathcal{S}})\right) = 2\left(\pi(1)^2\right) = 2\pi. \end{aligned}$$

Here ae some great uses for Stokes' Theorem:

(1) A surface is called *compact* if it is closed as a set, and bounded. A surface is called *closed* if it is compact and has no boundary.
   - Surfaces like the 2-sphere S^2, and the 2-torus T^2 are closed, while the disk, or a surface which is the continuous injective graph of a closed rectangle in the plane (we tend to call this a *(flying) carpet.*
   - Recall that a curve in $\mathbb{R}^n$ is *simple* if it does not intersect itself. Hence a bounded simple curve with its endpoints is *compact*. Its boundary are the two endpoints. A *closed* curve forms a loop, and hence has no boundary.
   - Also, the surface of teh unit cube in $\mathbb{R}^3$ is closed. Creases and corners are not considered boundaries of a surface. So, in a very mathematical sense, there is no real difference between the surface of a cube, and the surface of a ball. One does have edges and corners and the other does not, but each does enclose space. Now, there is a difference, though. The surface of the unit cube is not smooth, while the surface of the ball is. But we can still integrate over each.

The difference is that, to integrate over the cube (a piece-wise smooth surface), we would have to break up the integral calculation into each face, and then add the results at the end.

(2) In the case of a closed surface, the curl of <u>any</u> vector field in $\mathbb{R}^3$ over a closed surface will be 0, by order of Stokes' Theorem: The total flux of the curl of a vector field over a surface is equal to the circulation of that vector field over the boundary of the surface. If the surface has no boundary, then there is no circulation. Then by Stokes', the curl of $\mathbf{F}$ has no flux across the surface.

(3) In contrast, let $\mathbf{F}$ be a conservative vector field, so $\mathbf{F} = \nabla f$ for a real-valued, C^1-function. Then, for any surface $\mathcal{S}$ that satisfies Stokes', the circulation of $\mathbf{F}$ along $\partial\mathcal{S}$ is 0 (we say it vanishes), or

$$\int_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s} = 0.$$

Why is this? For any conservative vector field, $\nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0}$. Hence by Stokes'

$$\iint_{\mathcal{S}} \nabla \times \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{0} \bullet d\mathbf{S} = 0 = \oint_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s}.$$

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LECTURE 22: THE THEOREM OF GAUSS.

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Synopsis. In this lecture, we finish the foundational material of what makes a vector calculus course with a full discussion of the last of the Big Theorems, the theorem of Gauss. Really, the BIg Three theorems we are discussing are all similar in nature yet vary in dimension. Again, for Gauss' Theorem, we state the theorem and discuss its constituent pieces, develop the intuition needed to see what the theorem states, and establish some main situations where the theorem is relevant. Then we use Gauss' and Stokes' Theorems to give a much more precise idea of just what the divergence and the curl of a vector field actually is and how to understand these concepts geometrically.

22.1. **Gauss' Theorem.**

Theorem 22.1 (Gauss' Theorem)**.** *Let $\mathcal{W} \in \mathbb{R}^3$ be a solid region, whith $\partial\mathcal{W}$ a finite set of piecewise smooth, closed, orientable surfaces, oriented outwardly from $\mathcal{W}$. For $\mathbf{F}$ a C^1-vector field on a domain containing $\mathcal{W}$,*

$$\left(\oiint_{\partial\mathcal{W}} (\mathbf{F}\bullet\mathbf{n})\, dS\ =\right) \oiint_{\partial\mathcal{W}} \mathbf{F}\bullet d\mathbf{S} = \iiint_{\mathcal{W}} \nabla\bullet\mathbf{F}\, dV \left(= \iiint_{\mathcal{W}} (\mathbf{div}\,\mathbf{F})\, dV\right).$$

Special Notes:

(1) Recall that any compact domain in $\mathbb{R}^2$ with nonempty interior has a set of closed curves as boundary. In $\mathbb{R}^3$, any compact domain with nonempty interior has a set of closed surfaces as boundary. And this generalizes to higher dimensions readily. But this leads to the conclusion: *The boundary of a compact region with nonempty interior has no boundary!* Think about this.

(2) The proof of Gauss' Theorem is elementary and quite straightforward: Denoting points in $\mathbb{R}^3$ in the obvious way using the variable x, y, and z, and with $\mathbf{F} = F_1(x,y,z)\,\mathbf{i} + F_2(x,y,z)\,\mathbf{j} + F_3(x,y,z)\,\mathbf{k}$, we first see that

$$\iiint_{\mathcal{W}} (\mathbf{div}\,\mathbf{F})\, dV = \iiint_{\mathcal{W}} \frac{\partial F_1}{\partial x}\, dV + \iiint_{\mathcal{W}} \frac{\partial F_2}{\partial y}\, dV + \iiint_{\mathcal{W}} \frac{\partial F_3}{\partial z}\, dV.$$

Then, it should also be clear that

$$\iint_{\partial\mathcal{W}} (\mathbf{F}\bullet\mathbf{n})\, dV = \iint_{\partial\mathcal{W}} F_1\,\mathbf{i}\bullet\mathbf{n}\, dV + \iint_{\partial\mathcal{W}} F_2\,\mathbf{j}\bullet\mathbf{n}\, dV + \iint_{\partial\mathcal{W}} F_3\,\mathbf{k}\bullet\mathbf{n}\, dV.$$

And finally, one simply shows that each is equal to each, respectively.

Exercise 1. Show $\displaystyle\iiint_{\mathcal{W}} \frac{\partial F_1}{\partial x}\, dV = \iint_{\partial\mathcal{W}} F_1\,\mathbf{i}\bullet\mathbf{n}\, dV$, when $\mathcal{W}$ is elementary in all directions.

So we are now fully in a position to understand some concepts that we have previously only vaguely discussed:

- Divergence of a vector field.
- Curl of a vector field.

22.2. **Divergence.**

- Intuitive definition: Measures the infinitesimal expansion of volume under the flow of a vector field.
- Actual definition: Measures the aggregate flux of a vector field across the boundary of an infinitesimal ball centered at a point.

Theorem 22.2. *Let* $\mathbf{F}$ *be a* C^1*-vector field defined in some (open) neighborhood of a point* $\mathbf{p} \in \mathbb{R}^3$*. For*

$$\mathcal{S}_a = \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; ||\mathbf{x} - \mathbf{p}|| = a \right\},$$

a 2-sphere of radius $a > 0$ *centered at* $\mathbf{p}$ *and oriented outwardly,*

$$\mathbf{div}\,\mathbf{F}(\mathbf{p}) = \lim_{a \to 0^+} \frac{3}{4\pi a^3} \oiint_{\mathcal{S}_a} \mathbf{F} \bullet d\mathbf{S}.$$

Proof. For any $f \in C^0[\mathcal{W} \subset \mathbb{R}^3, \mathbb{R}]$, $\mathcal{W}$ a bounded solid region, there exists $\mathbf{q} \in \mathbb{R}^3$ where

$$\iiint_{\mathcal{W}} f(x, y, z)\, dV = f(\mathbf{q}) \cdot \mathbf{volume}(\mathcal{W}).$$

This is the Mean Value Theorem for triple integrals.

Now since $\mathbf{F}$ is C^1 in an open neighborhood of $\mathbf{p}$, there exists $\epsilon > 0$ such that $\mathbf{F}$ is C^1 on

$$\overline{\mathcal{B}}_\epsilon = \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; ||\mathbf{x} - \mathbf{p}|| \leq \epsilon \right\}.$$

Then $\epsilon = a$ in the theorem and $\mathcal{S}_a = \partial \overline{\mathcal{B}}_\epsilon$. Then, there exists a $\mathbf{q} \in \overline{\mathcal{B}}_\epsilon$ such that

$$\begin{aligned} \iiint_{\overline{\mathcal{B}_\epsilon}} \mathbf{div}(\mathbf{F})\, dV &= \mathbf{div}\left(\mathbf{F}(\mathbf{q})\right) \cdot \mathbf{volume}\left(\overline{\mathcal{B}}_\epsilon\right) \\ &= \frac{4\pi\epsilon^3}{3} \mathbf{div}\left(\mathbf{F}(\mathbf{q})\right), \end{aligned}$$

since divergence is simply a scalar field on $\overline{\mathcal{B}}_\epsilon$. Now, we can use Gauss' Theorem:

$$\begin{aligned} \lim_{\epsilon \to 0^+} \frac{3}{4\pi\epsilon^3} \oiint_{\mathcal{S}_\epsilon} \mathbf{F} \bullet d\mathbf{S} &\overset{\textbf{Gauss}}{=\!=} \lim_{\epsilon \to 0^+} \frac{3}{4\pi\epsilon^3} \iiint_{\overline{\mathcal{B}}_\epsilon} \mathbf{div}\left(\mathbf{F}\right)\, dV \\ &= \lim_{\epsilon \to 0^+} \frac{3}{4\pi\epsilon^3} \left(\frac{4\pi\epsilon^3}{3} \cdot \mathbf{div}\left(\mathbf{F}(\mathbf{q})\right) \right) \\ &= \lim_{\epsilon \to 0^+} \mathbf{div}\left(\mathbf{F}(\mathbf{q})\right) = \mathbf{div}\left(\mathbf{F}(\mathbf{p})\right). \end{aligned}$$

□

So Gauss' Theorem says that the amount of volume created or lost upon flowing along a vector field in a compact solid $\mathcal{W}$ is equal to the total amount flowing through the boundary $\partial\mathcal{W}$.

22.3. **Curl.**

- Intuitive definition: Measures the twisting effect of a vector field in $\mathbb{R}^3$ felt by flowing along it.
- Actual definition: Measures the total circulation of a vector field along an edge of an infinitesimal disk normal to the vector field at a point.

Theorem 22.3. *Let* $\mathbf{F}$ *be a* C^1*-vector field defined in some (open) neighborhood of a point* $\mathbf{p} \in \mathbb{R}^3$*. Let* $\mathbf{n}$ *be a unit vector based at* $\mathbf{p}$*, with*

$$D_a = \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; ||\mathbf{x} - \mathbf{p}|| \leq a, \; (\mathbf{x} - \mathbf{p}) \bullet \mathbf{n} = 0 \right\},$$

a 2-disk of radius $a > 0$ *centered at* $\mathbf{p}$ *and normal to* $\mathbf{n}$*. Orient* D_a *compatibly with* $\mathbf{n}$ *and also* $C_a = \partial D_a$*. Then the component of curl in the direction of* $\mathbf{n}$ *is*

$$\mathbf{curl}\,\mathbf{F}(\mathbf{p}) \bullet \mathbf{n} = \lim_{a \to 0^+} \frac{1}{\pi a^2} \oint_{C_a} \mathbf{F} \bullet d\mathbf{s}.$$

Hence, curl at a point is the infinitesimal circulation of $\mathbf{F}$ along a loop perpendicular to the direction of flow. In essence, choose a unit vector $\mathbf{n}$ based at $\mathbf{p}$, and form a small disk normal to $\mathbf{n}$ and containing $\mathbf{p}$. If the vector field generally points in a direction along the boundary of the disk compatibly with its orientation with the disk on its left (counterclockwise), then the circulation will be positive. If, in the aggregate, it points in the opposite direction to the orientation on the boundary of the disk, then the circulation will be negative. And if, in the aggregate, the vector field is orthogonal to the boundary of the disk, then the circulation will be 0.

Now allow $\mathbf{n}$ to vary. The magnitude of $\mathbf{curl}\,\mathbf{F}(\mathbf{p}) \bullet \mathbf{n}$ will be maximized at $\mathbf{p}$ precisely when

$$\mathbf{n} = \frac{\mathbf{curl}\,\mathbf{F}(\mathbf{p})}{||\mathbf{curl}\,\mathbf{F}(\mathbf{p})||}.$$

Therefore the twisting or rotating effect of the vector field $\mathbf{F}$ at $\mathbf{p}$ is greatest about the axis parallel to $\frac{\mathbf{curl}\,\mathbf{F}(\mathbf{p})}{||\mathbf{curl}\,\mathbf{F}(\mathbf{p})||}$. One can use this as a definition of the curl of a vector field.

Proof. Exactly like the previous theorem but using Stokes instead of Gauss. □

Notes:

- The quantity $\mathbf{div}\,(\mathbf{F}(\mathbf{p}))$ is also called the *flux density* of $\mathbf{F}$ at $\mathbf{p}$: It is the limit of the flux per unit volume.
- The quantity $\mathbf{curl}\,(\mathbf{F}(\mathbf{p}))$ is also called the *circulation density* of $\mathbf{F}$ at $\mathbf{p}$: It is the limit of the flux per unit volume.

Stokes' Theorem says that the total rotational effect of a vector field on a surface in $\mathbb{R}^3$ is equal to the aggregate boost or hindrance of a particle on the edge.

Green's Theorem is simply Stokes' Theorem limited to domains in the plane.

Example 22.4. ***For*** $\mathbf{F} = 2x\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ ***and***

$$\mathcal{S} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 = 1 \right\},$$

the unit 2-sphere in $\mathbb{R}^3$***, find the flux of*** $\mathbf{F}$ ***through*** $\mathcal{S}$***.***

The solution here is a calculation of

$$\oiint_{\mathcal{S}} (\mathbf{F} \bullet \mathbf{n})\, dS,$$

where $\mathbf{n}$ is the unit normal vector to $\mathcal{S}$. We use Gauss' Theorem to instead integrate the divergence of $\mathbf{F}$ on the unit ball

$$\overline{\mathcal{B}} = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2+y^2+z^2 \leq 1\right\}.$$

By Gauss,

$$\begin{aligned}
\oiint_{\mathcal{S}} (\mathbf{F} \bullet \mathbf{n})\, dS &= \iiint_{\overline{\mathcal{B}}} \mathbf{div}\,\mathbf{F}\, dV \\
&= \iiint_{\overline{\mathcal{B}}} 2(1+y+z)\, dV \\
&= 2\iiint_{\overline{\mathcal{B}}} dV + 2\iiint_{\overline{\mathcal{B}}} y\, dV + 2\iiint_{\overline{\mathcal{B}}} z\, dV.
\end{aligned}$$

We do the middle integral first: We have

$$\begin{aligned}
2\iiint_{\overline{\mathcal{B}}} y\, dV &= 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} y\, dy\, dz\, dx \\
&= 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \underbrace{\left(\frac{y^2}{2}\bigg|_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}}\right)}_{0} dz\, dx \\
&= 0.
\end{aligned}$$

The same will be true for the third integral. And so we have ,

$$\begin{aligned}
\oiint_{\mathcal{S}} (\mathbf{F} \bullet \mathbf{n})\, dS &\overset{\textbf{Gauss}}{=\!=\!=} \iiint_{\overline{\mathcal{B}}} \mathbf{div}\,\mathbf{F}\, dV \\
&= 2\iiint_{\overline{\mathcal{B}}} dV = 2\cdot \mathbf{vol}\left(\overline{\mathcal{B}}\right) = 2\left(\frac{4\pi(1)^3}{3}\right) = \frac{8\pi}{3}.
\end{aligned}$$