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LECTURE 20: SURFACE INTEGRATION.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. We continue with the idea of understanding how the calculus of functions behaves along parameterized surfaces (instead of along parameterized curves.) Today, we define and study both scalar and vector surface integrals, of real-valued functions and vector fields along surfaces embedded in three space, respectively. We sill stick to surfaces in three space for the expediency of understanding these concepts without too much intricate machinery. But we will allude regularly to the idea that we can embed and parameterize a surface in n-space, $(n > 2)$, and play the same game. We also discuss the idea of reparameterizations, orientation of a surface, and geometric interpretations, all as a lead up to another of the three big theorems, Stoke's Theorem.

20.1. **Functions defined on surfaces.** In a similar fashion that we integrate functions (real-valued) and vector fields (vector-valued) over curves, we can do so over surfaces:

- Like for scalar line integrals, if the surface in $\mathbb{R}^3$ lies inside the domain of a real-valued function on (a part of) $\mathbb{R}^3$, we can restrict the domain of the function to only the surface. Then adding up the values of the function, essentially integrating the function, over the surface is straightforward.
- If the surface is parameterized, so has coordinates defined directly on it, then the integration process can utilize the parameterization and we integrate using a double integral. However, it should also be obvious that the value of such an integration must be independent of any particular choice of parameterization.
- Basically, for a surface $\mathcal{S}$, we look to define the quantity
$$\iint_{\mathcal{S}} f\,dS, \quad \text{where} dS = ||\mathbf{N}||\;dA,$$
where, by the previous lecture, the vector $\mathbf{N}$ is the normal to the surface at a point, defined via the cross product of the tangent partial derivative vectors of the parameterization, and dA is the area-form given also by the parameterization.
- Given a parameterization $\mathbf{X} : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^n$, we will refer to the surface as simply $\mathbf{X}$ or $\mathbf{X}(s,t)$. This is entirely similar to our reference to a curve parameterization $\mathbf{x} : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}^n$ as simply $\mathbf{x}$, or $\mathbf{x}(t)$, when appropriate.

Definition 20.1. Let $\mathbf{X} : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth parameterized surface, with $\mathcal{D}$ bounded. Let f be a C^0 function defined on a domain that includes $\mathbf{X}$. Then the *scalar surface integral* of f along $\mathbf{X}$ is

$$\begin{aligned}\iint_{\mathbf{X}} f\,dS &= \iint_{\mathcal{D}} f\left(\mathbf{X}(s,t)\right) ||\mathbf{X}_s \times \mathbf{X}_t||\;ds\,dt \\ &= \iint_{\mathbf{X}} f\left(x(s,t), y(s,t), z(s,t)\right) \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}\;ds\,dt.\end{aligned}$$

Some notes:

- Like for line integrals, dS is a scalar 2-form (as ds is a scalar 1-form), and represents an infinitesimal change in surface area along the surface.
- For $f(x, y, z) = 1$, the scalar surface integral of f gives the surface area of $\mathbf{X}$.
- In the parameter coordinates (s, t), this looks like a standard double integral.
- If $\mathbf{X}$ is not smooth, but has edges, so is piecewise smooth, then each smooth piece must be integrated separately, and the results added together.

Definition 20.2. Let $\mathbf{X} : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth parameterized surface, with $\mathcal{D}$ bounded. Let $\mathbf{F}$ be a C^1-vector field defined on a domain that includes $\mathbf{X}(\mathcal{D})$. Then the *vector surface integral* of $\mathbf{F}$ along $\mathbf{X}$ is

$$\begin{aligned}\iint_{\mathbf{X}} \mathbf{F} \bullet d\mathbf{S} &= \iint_{\mathcal{D}} \mathbf{F}\left(\mathbf{X}(s,t)\right) \bullet \mathbf{N}(s,t)\, ds\, dt \\ &= \iint_{\mathbf{X}} \mathbf{F}\left(x(s,t), y(s,t), z(s,t)\right) \bullet \begin{bmatrix} \frac{\partial(y,z)}{\partial(s,t)} \\ -\frac{\partial(x,z)}{\partial(s,t)} \\ \frac{\partial(x,y)}{\partial(s,t)} \end{bmatrix} ds\, dt.\end{aligned}$$

Some notes:

- Here $d\mathbf{S} = \mathbf{N}(s, t)\, ds\, dt$ is a vector 2-form. It is the differential of surface area written in terms of the normal to the surface at (s, t).
- If we normalize the normal vector

  $$\mathbf{n}(s,t) = \frac{\mathbf{N}(s,t)}{||\mathbf{N}(s,t)||},$$

  then

  $$\begin{aligned}\iint_{\mathbf{X}} \mathbf{F} \bullet d\mathbf{S} &= \iint_{\mathcal{D}} \mathbf{F}\left(\mathbf{X}(s,t)\right) \bullet \mathbf{N}(s,t)\, ds\, dt \\ &= \iint_{\mathcal{D}} \mathbf{F}\left(\mathbf{X}(s,t)\right) \bullet \mathbf{n}(s,t)\, ||\mathbf{N}(s,t)||\; ds\, dt \\ &= \iint_{\mathbf{X}} \left(\mathbf{F} \bullet \mathbf{n}\right)\, dS.\end{aligned}$$

  Thus, the vector surface integral of a vector field along a surface is just the scalar surface integral of the component of the vec tor field normal to the surface, along the surface. <u>This concept wil be very important in the near future!</u>

So what is the geometric interpretation of a vector surface integral? The quantity $\iint_{\mathbf{X}} \mathbf{F} \bullet d\mathbf{S}$ measures the vector field flow <u>through</u> the surface. This is also called the *flux* of $\mathbf{F}$ through $\mathbf{X}$. Compare this to the interpretation of the vector line integral $\int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s}$, the circulation, which measures the vector field flow in the direction of $\mathbf{c}$ along $\mathbf{c}$.

Definition 20.3. Let $\mathbf{X} : \mathcal{D}_1 \subset \mathbb{R}^2 \to \mathbb{R}^3$ and $\mathbf{Y} : \mathcal{D}_2 \subset \mathbb{R}^2 \to \mathbb{R}^3$ be two parameterizations, such that $\mathbf{X}(\mathcal{D}_1) = \mathbf{Y}(\mathcal{D}_2)$. The $\mathbf{Y}$ is called a reparameterization of $\mathbf{X}$ if there exists a one-to-one and onto $\mathbf{H} : \mathcal{D}_2 \to \mathcal{D}_1$, with inverse $\mathbf{H}^{-1} : \mathcal{D}_1 \to \mathcal{D}_2$, such that $\mathbf{Y} = \mathbf{X} \cdot \mathbf{H}$

Note that a reparameterization is called smooth if both $\mathbf{X}$ and $\mathbf{Y}$ are smooth and if $\mathbf{H}$ is C^1. Here are some facts:

(1)

Theorem 20.4. *For f a C^0-function on a domain including a smooth $\mathbf{X} : \mathcal{D} \to \mathbb{R}^3$, then for any smooth reparameterization $\mathbf{Y}$ of $\mathbf{X}$,*

$$\iint_{\mathbf{Y}} f\, dS = \iint_{\mathbf{X}} f\, dS.$$

(2) For a smooth curve $\mathbf{c}$, an *orientation* is a choice of a continuously varying *unit tangent vector* along $\mathbf{c}$. This is entirely consistent with the idea of simply choosing a direction of travel along the curve. Note that an orientation is given automatically when one parameterizes the curve (the direction of increase in values of the parameter), but one may also "choose" to move in the opposite direction, as we did when proving Green's Theorem.

For a smooth surface $\mathbf{X}$, an *orientation* is a choice of continuously varying *unit normal vector* along $\mathbf{X}$. In effect, you are choosing a "side", as in above vs. below, or inside vs. outside. Note that there are surfaces that are not orientable. But note that any parameterization $\mathbf{X}$ automatically orients the surface, since

$$\mathbf{n}(s,t) = \frac{\mathbf{N}(s,t)}{||\mathbf{N}(s,t)||} = \frac{\mathbf{X}_s(s,t) \times \mathbf{X}_t(s,t)}{||\mathbf{X}_s(s,t) \times \mathbf{X}_t(s,t)||}.$$

(3) One may ask if two smooth parameterizations $\mathbf{X}$ and $\mathbf{Y}$ defining the same surface are oriented compatibly. Essentially, we are asking if a smooth reparameterization *preserves orientation*. The answer is yes, if the two calculated unit normal vectors at each point are "on the same side". But this can be determined by $\mathbf{H} : \mathcal{D}_2 \to \mathcal{D}_1$, the function defining the reparameterization. Indeed, let $\mathbf{Y}(s,t)$, defined on $\mathcal{D}_2$ be a reparameterization of $\mathbf{X}(u,v)$, defined on the domain $\mathcal{D}_1$, with $\mathbf{H}$, defined as above. Then $\mathbf{Y}(s,t) = \mathbf{X}(u,v) = \mathbf{X}\left(\mathbf{H}(s,t)\right)$. Then one can use the Chain Rule to show

$$\mathbf{N}_{\mathbf{Y}}(s,t) = \frac{\partial(u,v)}{\partial(s,t)} \mathbf{N}_{\mathbf{X}}(u,v),$$

where the subscripts on the normal vectors here only denote to which parameterization the normal vector belongs. Hence if the Jacobian determinant of $\mathbf{H}$ is positive, the reparameterization is *orientation preserving*. If negative, then the reparameterization is *orientation reversing*: We have

Theorem 20.5. *For $\mathbf{F}$ a C^0-vector field on a domain including a smooth $\mathbf{X} : \mathcal{D} \to \mathbb{R}^3$, then for any orientation preserving smooth reparameterization $\mathbf{Y}$,*

$$\iint_{\mathbf{Y}} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathbf{X}} \mathbf{F} \bullet d\mathbf{S}.$$

If $\mathbf{Y}$ is orientation reversing, then

$$\iint_{\mathbf{Y}} \mathbf{F} \bullet d\mathbf{S} = -\iint_{\mathbf{X}} \mathbf{F} \bullet d\mathbf{S}.$$

(4) Orienting a surface automatically orients the boundary of the surface. Indeed, let $\mathcal{S}$ be an oriented surface with boundary in $\mathbb{R}^3$ such that $\partial\mathcal{S}$ is a piecewise-C^1 closed curve. Let $\mathbf{p} = (s_0, t_0) \in \partial\mathcal{S}$, where

$$\mathbf{p} = (s_0, t_0) = \left(x(s_0, t_0), y(s_0, t_0), z(s_0, t_0)\right),$$

and choose $\mathbf{c} : [a, b] \to \mathcal{S} \subset \mathbb{R}^3$ a smooth curve in the surface so that $\mathbf{c}(a) = \mathbf{p}$ and $\mathbf{c} \cap \partial\mathcal{S} = \{\mathbf{p}\}$. Now define

$$\mathbf{n}(\mathbf{p}) = \lim_{t\to a} \mathbf{n}\left(\mathbf{c}(t)\right), \quad \text{and} \quad \mathbf{v}(a) = \lim_{t\to a} \mathbf{c}'(t).$$

Here, $\mathbf{n}$ and $\mathbf{v}$ are vectors based at $\mathbf{p}$ and are orthogonal to each other. Hence they determine a two dimensional plane in $\mathbb{R}^3$ as the set of all linear combinations. Then $\mathbf{n} \times \mathbf{v}$ is perpendicular to both, and using the right-hand rule, determines a unique direction on $\partial\mathcal{S}$.

This is the direction specified in Green's Theorem!