January 30, 2019

LECTURE 2: FUNCTIONS OF SEVERAL VARIABLES.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. Today we begin the course in earnest in Chapter 2, although, again like on Monday, we will be covering the material mostly for notation and viewpoint. Pay close attention to why and how we visualize functions, though parameterizations, graphs, slices and sections. These will expose the visual clues to how we analyze functions.

Helpful Documents.

- Mathematica: `CurvesInSpace`,
- Mathematica: `ParameterizedSurfaces`,
- Mathematica: `VisualizingFunctions`, and
- PDF: LevelSets.

Functions of Several Variables. A function $f : X \to Y$ from a set X to another set Y is defined in a manner equal to what you have already studied in single variable calculus (and pre-calculus):

- f assigns to each $x \in X$ a single element $y \in Y$.
- The set X is called the *domain* of the function, and Y is called the *codomain*.
- $f(X) \subset Y$ (as a set) is called the *range* of f, and more precisely called the *image* of X in Y under f. It can be defined as
$$f(X) = \left\{ y \in Y \;\middle|\; y = f(x) \text{ for some } x \in X \right\}.$$
- For a subset $Z \subset Y$, the set
$$f^{-1}(Z) = \left\{ x \in X \;\middle|\; f(x) \in Z \right\}$$
is called the *inverse image* of Z in X under f, or the *preimage* of Z in X (under f). Note that $f^{-1}(y) = \emptyset$, when $y \notin f(X)$.
- f is called *one-to-one*, or *injective*, if
$$\#\left\{ x \in X \;\middle|\; f(x) = y \right\} \leq 1, \quad \forall y \in Y.$$
- f is called *onto* or *surjective* if $\forall y \in Y$, $y = f(x)$ for at least one $x \in X$.
- f is called *bijective* if f is both injective and surjective.

Note that, for this class, X and Y will be subsets of Euclidean space, although often not the same space nor the same dimension.

Here is some additional nomenclature and notation:

- Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. If $m = 1$, we call $f : X \to Y$ a *real-valued* or *scalar-valued* function on X, or on n-variables (restricted to X). If $m > 1$, we say f is *vector-valued*. As we will see, vector-valued functions consist of expressions that are real-valued on each coordinate of $\mathbb{R}^m$.

- Where important to the discussion, we will denote scalars as $x \in \mathbb{R}$, and vectors as $\mathbf{x} \in \mathbb{R}^n$, $n > 1$. We will also denote a real-valued function as f, and a vector-valued function as $\mathbf{f}$. In lecture, we will employ the vector notation $\vec{x}$ and $\vec{f}$, since boldface is difficult in chalk. Note that when it is not important to the discussion, or for general situations, it is the case that we will use boldface for variables, and possibly write $f : X \to Y$, and $f(x) = y$, even if $X \in \mathbb{R}^n$, $n > 1$, and $y \in Y \subset \mathbb{R}^m$, $m > 1$. This is common in analysis and should be clear in context.
- a function $f : X \to Y$ is often called a *map* (or a *mapping*) from X to Y. In some contexts, a function and a map are not the same thing, but often they are used interchangeably.

  Definition 2.1. A map $p : X \to X$ is called a *projection* if $p(p(x)) = p(x)$, $\forall x \in X$.

  - Here, the set comprising the image $p(X) \subset X$ is called the *projection* of X onto $p(X)$. When X is a linear space and p a linear projection, then $p(X)$ is a linear subspace. See Example 2.2 below.
  - A projection p, restricted to its image, is the *identity map*. We can write this as $p\big|_{p(X)} = Id_{p(X)}$.
  - For $X = \mathbb{R}^n$, the map $p_i : \mathbb{R}^n \to \mathbb{R}^n$, defined by

    $$p_i\left((x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)\right) = (0, \ldots, 0, x_i, 0, \ldots, 0)$$

    is called the *ith projection*. Sometimes, one may write $p_i(\mathbf{x}) = x_i$, but this is not quite correct.
  - There are many extensions and generalizations of the idea of projection in various areas of mathematics, including some that do not seem to fit the definition above. (See, for instance, the separate document `StereographicProjection`.) For now, here are a couple of examples.

Example 2.2. A common projection onto a linear subspace of $\mathbb{R}^n$ is to zero out one or more coordinates: In $\mathbb{R}^3$, the map

$$(x, y, z) \longmapsto (x, y, 0)$$

is a projection of 3-space onto the xy-plane (See the left side of Figure 2.1.

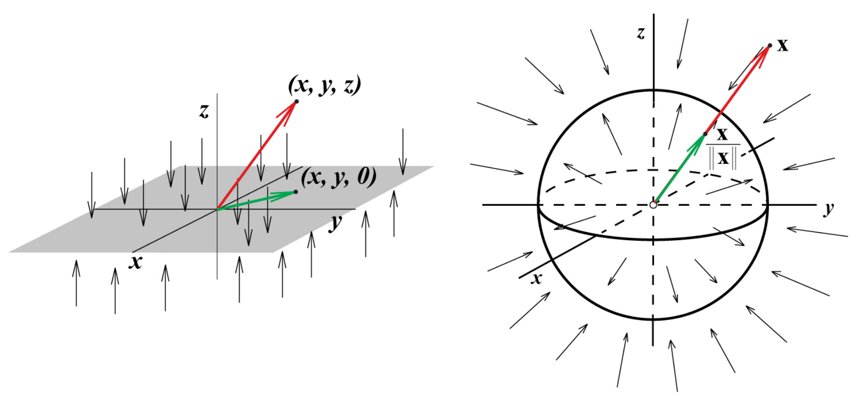

FIGURE 2.1. Projections in $\mathbb{R}^3$ onto the xy-plane (at left), and the unit sphere S^2 (at right).

Example 2.3. The map $r : \mathbb{R}^3 - \{\mathbf{0}\} \to S^2$, $r(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||}$ is the map which normalizes every non-zero vector in $\mathbb{R}^3$. Here

$$S^2 = \left\{\mathbf{x} \in \mathbb{R}^3 \;\middle|\; ||\mathbf{x}|| = 1\right\}$$

is called the *unit sphere* in $\mathbb{R}^3$, seen on the right of Figure 2.1. Do you see why $\mathbf{0} \in \mathbb{R}^3$ cannot be in the domain of r?

Visualizing functions either defined on subsets of $\mathbb{R}^n$ and/or to $\mathbb{R}^n$, when $n > 1$ can be tricky. Some tools that are useful include:

Graphs - In its most basic form, a *relation* is defined as any subset of the Cartesian product of two (or more) sets. And then a *graph* of a relation is just any visual depiction of that relation. When the two sets are subsets of real space $X \in \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, then the relation is a subset of $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$. Often, relations among real variables are given by equations, and in this case, the graph is the set of solutions to the equations "living" inside the direct product of copies of $\mathbb{R}$, one for each of the variables. And sometimes these relations are functional in one or more of the variables. In this case, solving the equation for one of the variables creates a function whose output is that solved-for variable and whose input(s) are the other variables. In this case, the graph of that function takes on a particular look; that of a"height over a floor" schematic:

Definition 2.4. For $f : X \subset \mathbb{R}^n \to \mathbb{R}$, the graph of f is the set

$$\mathbf{graph}(f) = \left\{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \;\middle|\; x_{n+1} = f(\mathbf{x})\right\}.$$

Note that this is quite useful for $n = 2$ (so that the graph "lives" in $\mathbb{R}^3$, but not so useful for $n > 2$. Also, this is the proper generalization for the way graphs of functions were constructed in pre-calculus and single variable calculus. And, generally speaking, the "size" of $f(X) \in \mathbb{R}^3$ will be the same as that of X. It should be easy to see that it is always the case that $\mathbf{graph}(f) \subset \mathbb{R}^{n+1}$ always projects to (a copy of) $X \subset \mathbb{R}^n \times \mathbb{R}$ as

$$(x_1, x_2, \ldots, x_n, f(\mathbf{x})) \longmapsto (x_1, \ldots, x_n, 0).$$

See Figure 2.2. More generally, we have:

Definition 2.5. For $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}^m$, $m \geq 1$, where $\mathbf{f}(\mathbf{x}) = \mathbf{y}$, the graph of $\mathbf{f}$ is the set

$$\mathbf{graph}(\mathbf{f}) = \left\{(\mathbf{x}, \mathbf{f}(\mathbf{x})) \in \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m} \;\middle|\; \mathbf{y} = \mathbf{f}(\mathbf{x})\right\}.$$

Consider the vector-valued function $\mathbf{g} : X \subset \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\mathbf{g}(\mathbf{x}) = \mathbf{g}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$. Here, for $i = 1, 2$, each $g_i : X \to \mathbb{R}$ is a real-valued function, called a *component function* or a *coordinate function*. But the graph of $\mathbf{g} \subset \mathbb{R}^4$ is the set

$$\mathbf{graph}(\mathbf{g}) = \left\{(x, y, z, u) \in \mathbb{R}^4 \;\middle|\; \begin{array}{c} z = g_1(x,y) \\ u = g_2(x,y) \end{array}\right\}.$$

It is already hard to visualize!

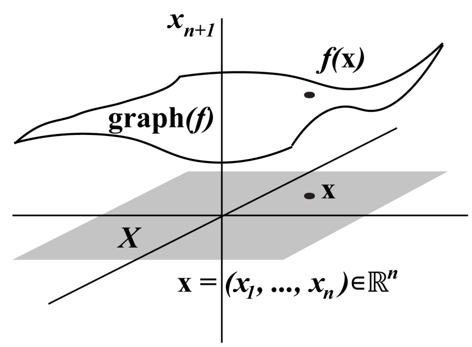

FIGURE 2.2. For $f : X \subset \mathbb{R}^n \to \mathbb{R}$, $\mathbf{graph}(f) \subset \mathbb{R}^{n+1}$.

An easier example to visualize is the function $\mathbf{h} : \mathbb{R} \to \mathbb{R}^2$, $\mathbf{h}(t) = (\cos t, \sin t)$. Its graph lives in $\mathbb{R}^3$ as a curve

$$\mathbf{graph}(\mathbf{h}) = \left\{ (t, x, y) \in \mathbb{R}^3 \;\middle|\; x = \cos t, y = \sin t \right\}.$$

As one can see in Figure 2.3, this curve can be visualized and studied, but is still a bit tricky to analyze.

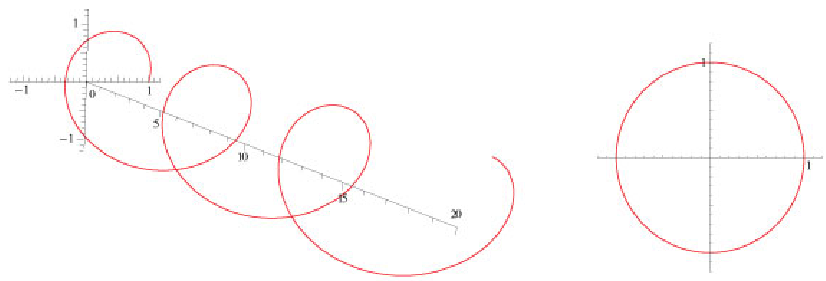

FIGURE 2.3. Projections in $\mathbb{R}^3$ onto the xy-plane (at left), and the unit sphere S^2 (at right).

Parameterizations - Generalized coordinates can be placed directly on a subset of $\mathbb{R}^n$ through continuous functions so that points on the subset are distinguishable via parameter values instead of ambient coordinates. (One does this on a sphere when one speaks of the latitude and longitude of a point on our Earth.) A *parameterization* allows one to describe a subset of $\mathbb{R}^n$ by a smaller number of variables; one can generally talk of a subset having a dimension equal to the number of variables it takes to distinguish points on the subset, although the notion of dimension for a space is not always very well defined.

Return to the function $\mathbf{h} : \mathbb{R} \to \mathbb{R}^2$, $\mathbf{h}(t) = (\cos t, \sin t) \in \mathbb{R}^2$, and consider only the image of $\mathbf{h} \subset \mathbb{R}^2$. Here, we say that h parameterizes the unit circle in the plane. In this case, t is a coordinate, defined directly (and only) on the circle of radius 1 in $\mathbb{R}^2$, and is a 1-dimensional parameterization. Note here that, broadly speaking, parameterizations should be one-to-one as functions, so that points are distinguished adequately. However, this is not true in general, and this example is telling. Here, we would say that this parameterization is locally-injective. We caution, though, that even this is not true in general.

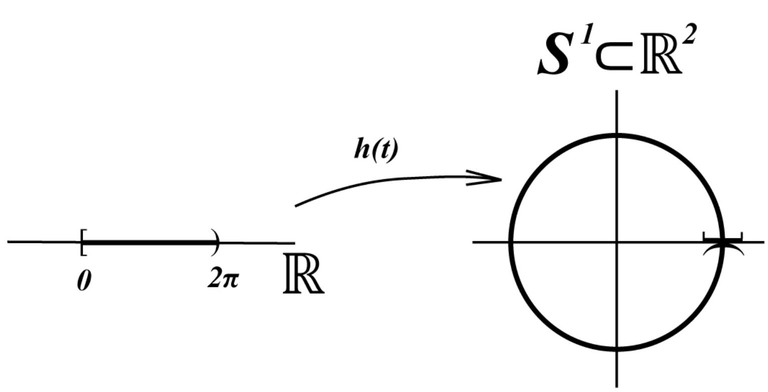

FIGURE 2.4. Parameterization of $S^1 \subset \mathbb{R}^2$ via $h : \mathbb{R} \to \mathbb{R}^2$, $h(t) = (\cos t, \sin t)$.

Example 2.6. Let $D \subset \mathbb{R}^2$ be the rectangle

$$D = \left\{ (\theta, \psi) \in \mathbb{R}^2 \;\middle|\; \theta \in [0, 2\pi],\ \psi \in [0, \pi] \right\}$$

as a subset of $\mathbb{R}^2$. Then the function $\Phi : D \to \mathbb{R}^3$, $\Phi(\theta, \psi) = (\sin\theta \sin\psi, \cos\theta \sin\psi, \cos\psi)$ provides coordinates directly on the unit sphere in three space that correspond to the azimuth angle θ and polar angle ψ of the standard spherical coordinate system in $\mathbb{R}^3$.

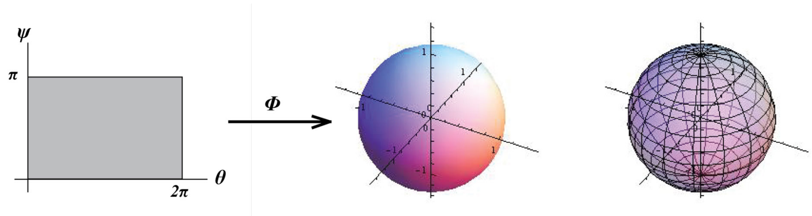

FIGURE 2.5. Parameterization of $S^2 \subset \mathbb{R}^3$ via $\Phi : D \to \mathbb{R}^3$.

Note here two things:

(1) The function Φ in Example 2.6 is injective, but only on the interior of D, and maps the bottom and top edges of D to the north and south poles, respectively, and maps both the left and right edges of D on top of each other and to one of the half-great circles stretching from the north pole to the south. Can you draw this seam on the sphere in Figure 2.5.

(2) The parameterized objects in these examples are not graphs of functions. They are visual depictions, yes, but they do not satisfy Definition **??**. They are actually the image of the function defining the parameterization. Hence we would call the unit circle $S^1 = \mathbf{image}(\mathbf{h})$ in Figure 2.4, and the two-sphere $S^2 = \mathbf{image}(\Phi)$ in Example 2.6.

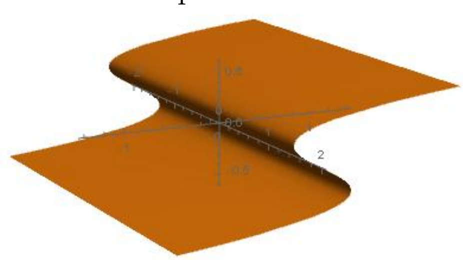

FIGURE 2.6. $\mathbf{image}(F) \subset \mathbb{R}^3$ in Example 2.7 is not the graph of a function defined on the xy-plane in $\mathbb{R}^3$.

Now the domain of the graph of a function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ always parameterizes $\mathbf{graph}(f) \subset \mathbb{R}^{n+1}$. See Figure 2.2. Can you see why? However, as seen with the sphere in Figure 2.5, it can also parameterize subsets of $\mathbb{R}^n$ that are not the graphs of functions:

Example 2.7. Let $R = [-2, 2] \times [-1.5, 1.5]$, and $F : R \to \mathbb{R}^3$, where $F(u, v) = \left(u, \frac{3(v^3 - v)}{4}, \frac{2v}{5} \right)$, shown in Figure 2.6.

Slices and Sections of graphs of functions - Understanding the features of graphs of functions of more than one variable can sometimes be facilitated by slicing through the graph, thus fixing the value of one or more variables, either parallel to the domain (a section), or perpendicular to the domain (a slice). First, some definitions:

Definition 2.8. Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a real-valued function on X.

(a) A *c-level set* of f is $\left\{ \mathbf{x} \in X \;\middle|\; f(\mathbf{x}) = c \right\}$.

(b) A *(horizontal) section* of f at c is the set $\left\{ (\mathbf{x}, c) \in \mathbf{graph}(f) \subset \mathbb{R}^{n+1} \;\middle|\; c = f(\mathbf{x}) \right\}$. Note that this is just the graph of a c-level set, and is sometimes called a *c-contour set.*

It is important to note here that a c-level set of a function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ is a subset of the domain X, while a horizontal section is a subset of the graph of X under f.

Example 2.9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 + y^2$. This function is called a *parabolic bowl* due to the shape of the graph, as in Figure 2.7, at left.

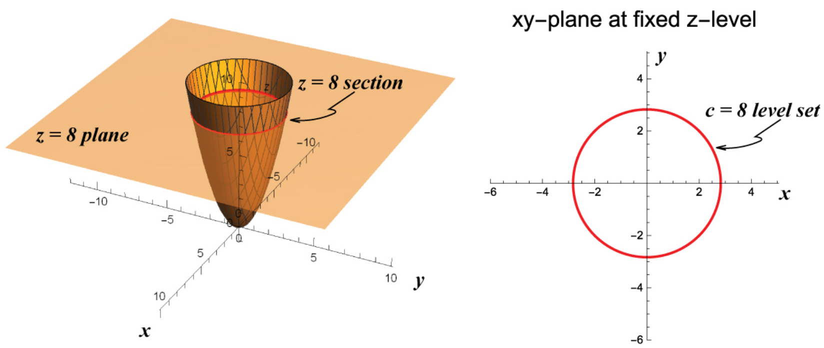

FIGURE 2.7. The $z = 8$-section and $c = 8$-level set of $z = f(x, y) = x^2 + y^2$.

Some notes:

- All c-level sets in Example!2.9 are circles in the plane, centered at the origin and of radius $\sqrt{c}$. They satisfy the equation $c = x^2 + y^2$.
- One can view c-level sets as the projections of a horizontal section back down into the domain, viewed as part of $\mathbb{R}^{n+1}$ representing the floor $z = 0$.
- One can also write a c-level set as the *inverse image* (as a set) of an output value $c \in \mathbb{R}$. Here $f^{-1}(c) \subset X$. Note that in Example 2.7, $f^{-1}(c) = \emptyset$, for $c < 0$. But still, $f^{-1}(c)$ is well-defined in these cases and $f^{-1}(-3) \subset X$, even as it is empty.

In contrast, a vertical section (or slice) of $\mathbf{graph}(f)$ is the intersection of $\mathbf{graph}(f)$ with a vertical subspace of $\mathbb{R}^{n+1}$ formed by setting one of the domain variables to a constant. So for

$$\mathbf{graph}(f) = \left\{ (x_1, x_2, \ldots, x_n, z) \in \mathbb{R}^{n+1} \;\middle|\; z = f(\mathbf{x}) \right\},$$

the x_i-slice at $x_i = c$ is the set

$$\left\{ (\mathbf{x}, z) \in \mathbb{R}^{n+1} \;\middle|\; z = f(x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_n) \right\}.$$

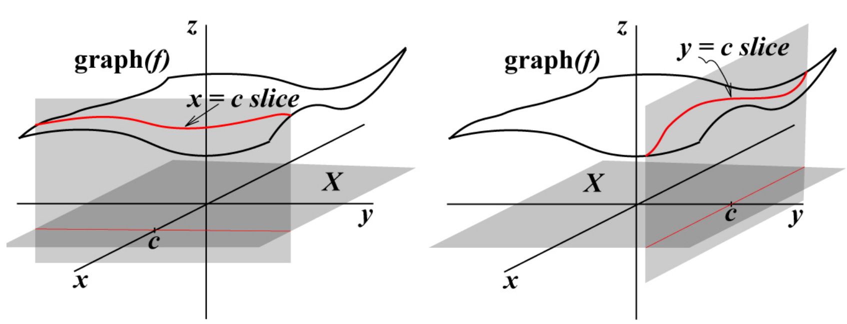

FIGURE 2.8. The 2 coordinate slices through $\mathbf{graph}(f)$, for $f : X \subset \mathbb{R}^2 \to \mathbb{R}$.

Back to $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + y^2$ in Example 2.9, the $y = 2$-slice is shown in Figure 2.9, and given by the equation $z = x^2 + 4$ in the xz-plane defined at $y = 2$. The $y = 2$-slice is a curve in $\mathbb{R}^3$ parameterized by x, and is the set

$$\left\{(x, 2, x^2 + 4) \in \mathbb{R}^3\right\}.$$

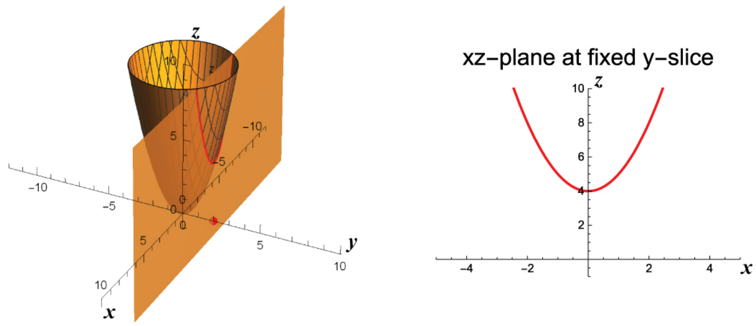

FIGURE 2.9. The y-slice at $y = 2$ of the function $z = f(x,y) = x^2 + y^2$.