April 10, 2019

LECTURE 19: SURFACE PARAMETERIZATIONS.

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Synopsis. The two dimensional counterpart to a curve in n-space is a surface in n-space, and today we define and discuss the properties of parameterized surfaces in (mostly) three-space (and sometimes n-space.) The parallels to curves will be obvious, and discussing these parallels will bring up very interesting contrasts, which we will highlight. Then we will begin the discussion of how a parameterization of a surface in space allows us to discuss the properties of the surface, including how functions behave when defined along the surface.

Helpful Documents. Mathematica: `ParameterizedSurfaces`.

19.1. **Coordinates on a surface.** We typically parameterize a curve in $\mathbb{R}^n$ via a map $\mathbf{c} : \mathcal{I} \to \mathbb{R}$, where $\mathcal{I} = [a, b] \subset \mathbb{R}$, and

$$\mathbf{c}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n.$$

For $n = 3$, we would usually write the three coordinate functions as $x(t)$, $y(t)$, and $z(t)$, all real-valued functions on $\mathcal{I}$.

Now let $\mathcal{D} \subset \mathbb{R}^2$ be a connected, open set, along with some or all of its boundary points. A parameterized surface in $\mathbb{R}^n$ is then a C^0-function $\mathbf{X} : \mathcal{D} \to \mathbb{R}^n$ $(n > 2)$, that is one-to-one on the interior of $\mathcal{D}$. The corresponding image of $\mathbf{X}$ is then

$$\mathbf{X}(s,t) = \begin{bmatrix} x_1(s,t) \\ x_2(s,t) \\ \vdots \\ x_n(s,t) \end{bmatrix} \in \mathbb{R}^n.$$

Example 19.1. Let $\mathcal{D} = \left\{(s,t) \in \mathbb{R}^2 \,\middle|\, 0 \le s \le 2\pi,\ 0 \le t \le \pi\right\} = [0, 2\pi] \times [0, \pi]$ in the st-plane. Then the map $\mathbf{X} : \mathcal{D} \to \mathbb{R}^3$, defined by

$$\mathbf{X}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} a\cos s \sin t \\ a \sin s \sin t \\ a \cos t \end{bmatrix}$$

has image S_a^2 the sphere of radius a centered at the origin in $\mathbb{R}^3$. In this parameterization, every point on the interior of $\mathcal{D}$ is mapped uniquely to a point on $S_a^2 = \mathbf{X}(\mathcal{D})$. However, the top and bottom edges of $\mathcal{D}$ are each all mapped to a point (the north and south poles, respectively, while the two walls are together mapped to a line from the north pole to the south pole. Perhaps we can call that line the "seam" of the ball? See Figure **??**.

Note: Just as a curve sits inside $\mathbb{R}^n$, $n > 2$, a surface can have $\mathbb{R}^n$, $n > 3$ as a codomain. For now, we will restrict our examples to three space $\mathbb{R}^3$ for the convenience and clarity of visualization.

Example 19.2. ***Graphs of functions are parameterizations.*** Any $f : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}$ has $\mathbf{graph}(f) \subset \mathbb{R}^3$ parameterized directly by the coordinates of $\mathcal{D}$, where $(x, y) \mapsto (x, y, f(x, y))$. The points in the domain parameterize $\mathbf{graph}(f) \subset \mathbb{R}^3$ because the third coordinate is uniquely specified once the first two are known.

Example 19.3. ***The square torus.*** The *2-torus* T^2 has a nice description as the product of two copies of the circle $T^2 = S^1 \times S^1$, where each coordinate is an angular one. This is fundamentally different form the two angular coordinates that comprise the 2-sphere, though. Here, let $\mathcal{D} = [0, 2\pi] \times [0, 2\pi] = [0, 2\pi]^2$. Then, for $a > b > 0$ positive constants, the map $\mathbf{X} : \mathcal{D} \to \mathbb{R}^3$, defined by

$$\mathbf{X}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} (a + b\cos t)\cos s \\ (a + b\cos t)\sin s \\ b\sin t \end{bmatrix}$$

has the shape of the surface of a doughnut. Here b is the cross-sectional radius and a is the radial distance from the z-axis to the center of any corss-sectional circle. The torus T^2 is often described as simply the unit square in the plane with the "opposite sides identified".

Definition 19.4. A surface $\mathcal{S} = \mathbf{X}(\mathcal{D})$ is *differentiable* if its coordinate functions are. When this is the case, then

$$\mathbf{X}_s(s_0, t_0) = \frac{\partial \mathbf{X}}{\partial s}(s_0, t_0) = \begin{bmatrix} \frac{\partial x}{\partial s}(s_0, t_0) \\ \frac{\partial y}{\partial s}(s_0, t_0) \\ \frac{\partial z}{\partial s}(s_0, t_0) \end{bmatrix}.$$

The same is true for $\mathbf{X}_t(s_0, t_0)$.

If these two first partials are continuous functions, then the derivative matrix $D\mathbf{X} = [\mathbf{X}_s, \mathbf{X}_t]$ exists and is a 3×2 matrix. Each of $\mathbf{X}_s$ and $\mathbf{X}_t$ is a vector of functions, and when evaluated at the point (s_0, t_0), each represents a vector tangent to the embedded surface $\mathcal{S} = \mathbf{X}(\mathcal{D})$, at the point $\big(x(s_0, t_0), y(s_), t_0), z(x_0, t_0)\big)$. Hence $\mathbf{X}_s(s_0, t_0)$ and $\mathbf{X}_t(s_0, t_0)$ are member of the tangent space to $\mathcal{S}$ at (s_0, t_0), where s, t are parameter coordinates on the surface, and $(x(s_0, t_0), y(s_0, t_0, z(s_0, t_0))$ is the corresponding point in the ambient coordinates in $\mathbb{R}^3$.

Now as $\mathbf{X}_s$ and $\mathbf{X}_t$ are always members of the tangent space to $\mathbf{X}(\mathcal{D}) \subset \mathbb{R}^3$, then $\mathbf{X}_s \times \mathbf{X}_t$ is normal to the surface (when it is nonzero and the surface is C^1, that is). Call this normal vector $\mathbf{N}(s_0, t_0) = (\mathbf{X}_s \times \mathbf{X}_t)(s_0, t_0)$.

Definition 19.5. The surface $\mathcal{S} = \mathbf{X}(\mathcal{D})$ is called *smooth at the point* $\mathbf{X}(s_0, t_0)$ if $\mathbf{X}$ is C^1 in an open neighborhood of (s_0, t_0) and if $\mathbf{N}(s_0, t_0) \neq \mathbf{0}$. The surface $\mathcal{S}$ is called *smooth* if it is smooth everywhere.

Note that C^1 and smoothness ensure that the embedded surface has no sharp edges only if the normal vector is nonzero. This is similar to the image of a parameterized curve, where if the parameter function is differentiable, one may still have a corner in the image of the curve if the tangent vector is $\mathbf{0}$.

Example 19.6. For $S_a^2 = \mathbf{X}(\mathcal{D})$, given in Example **??** above, we have

$$\mathbf{X}_s = \begin{bmatrix} -a\sin s \sin t \\ a\cos s \sin t \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{X}_t = \begin{bmatrix} a\cos s \cos t \\ a\sin s \cos t \\ -a\sin t \end{bmatrix}.$$

Hence $\mathbf{N} = \mathbf{X}_s \times \mathbf{X}_t = -a\sin t \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Hence, under this parameterization, $\mathbf{N}$ is nonzero everywhere except for when $t = 0, \pi$, so everywhere except for at the poles.

Exercise 1. Do the calculation that shows that $\mathbf{N} = \mathbf{X}_s \times \mathbf{X}_t = -a\sin t \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in Example **??** above.

However, S_a^2 is smooth everywhere, even at the poles. But not according to this parameterization. To see the poles as smooth, one would have to reparameterize so that the points corresponsding to the poles lie somewhere inside the corresponding region domain $\mathcal{D}$. In a sense, the parameterization we specified does give meaning to the phrase "One cannot walk east or west when standing at the north pole. One can only walk south!"

Definition 19.7. A *piecewise smooth* parameterized surface is the union of images of finitely many parameterized surfaces $\mathbf{X}_i : \mathcal{D}_i \to \mathbb{R}^3$, where each $\mathcal{D}_i$ is (1) elementary, (2) C^1 except possibly along $\partial\mathcal{D}_i$, and (3) each $\mathcal{S}_i = \mathbf{X}_i(\mathcal{D}_i)$ is smooth except at possibly a finite set of points.

19.2. **Surface area.** Recall that the length of a parameterized curve $\mathbf{c} : [a, b] \to \mathbb{R}^n$ can be calculated using the parameterization

$$\int_a^b ||\mathbf{c}'(t)||\, dt,$$

even as the actual length of the curve in $\mathbb{R}^n$ is independent of the parameterization. In 2-dimensions, we can develop a similar construction: Given a surface parameterization $\mathbf{X} : \mathcal{D} \to \mathbb{R}^n$, for a region $\mathcal{D}$, a small rectangle $\mathcal{R} \subset \mathcal{D}$, based at a point (s_0, t_0) and of size Δs and Δt, will have image $\mathbf{X}(\mathcal{R})$ a small region inside the surface $\mathcal{S} = \mathbf{X}(\mathcal{D})$. This image will most likely not be a parallelogram. But it can be approximated by one with sides $\mathbf{X}_s(s_0, t_0)\Delta s$ and $\mathbf{X}_t(s_0, t_0)\Delta t$. Then, the area of this image region is

$$\begin{aligned} \mathbf{area}\,(\mathbf{X}(\mathcal{R})) &\approx ||\mathbf{X}_s(s_0, t_0)\Delta s \times \mathbf{X}_t(s_0, t_0)\Delta t|| \\ &= ||\mathbf{X}_s(s_0, t_0) \times \mathbf{X}_t(s_0, t_0)||\, \Delta s\Delta t = ||\mathbf{N}(s_0, t_0)||\, \Delta s\Delta t. \end{aligned}$$

Note that this quantity is the area of the unit square inside the tangent space to $\mathcal{S}$ at (s_0, t_0), suitably scaled by Δs and Δt.

In the limit, as $\Delta s, \Delta t \to 0$, we get $||\mathbf{X}_s \times \mathbf{X}_t||\, ds\, dt$. With the idea that $\mathbf{area}(\mathcal{D}) = \iint_{\mathcal{D}} dA$, we get

$$\mathbf{area}(\mathcal{S}) = \iint_{\mathcal{S}} dS = \iint_{\mathcal{D}} ||\mathbf{X}_s \times \mathbf{X}_t||\, ds\, dt = \iint_{\mathcal{D}} ||\mathbf{N}(s, t)||\, ds\, dt.$$

Here $dS = ||\mathbf{N}(s, t)||\, dA$ is the differential of area, or an *area form* on $\mathcal{S}$.

Some notes:

- The expression $dS = ||\mathbf{N}(s,t)||\ dA$ is the 2-dimensional analog to $dc = ||\mathbf{c}'(t)||\ dA$ in the scalar-line integral.
- For $\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t)) \in \mathbb{R}^3$,

$$\mathbf{X}_s \times \mathbf{X}_t = \begin{bmatrix} \frac{\partial(y,z)}{\partial(s,t)} \\ -\frac{\partial(x,z)}{\partial(s,t)} \\ \frac{\partial(x,y)}{\partial(s,t)} \end{bmatrix},$$

so

$$\mathbf{area}(\mathcal{S}) = \iint_{\mathcal{D}} \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}\, ds\, dt,$$

where each of the summands under the radical is the square of a Jacobian determinant. Compare this to the calculation of arclength for a parameterized curve in the plane in single variable calculus: Given $\mathbf{c}(t) = (x(t), y(t))$,

$$\mathbf{arclength}(\mathbf{c}) = \int_{\mathbf{c}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 19.8. ***The surface area of*** S_a^2 ***a 2-sphere of radius*** a**.** Recall that $\mathbf{X}(s,t) = (a\cos s \sin t, a \sin s \sin t, a\cos t)$. So, as detailed in Example **??** above.

$$\mathbf{X}_s = \begin{bmatrix} -a\sin s\sin t \\ a\cos s\sin t \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{X}_t = \begin{bmatrix} a\cos s\cos t \\ a\sin s\cos t \\ -a\sin t \end{bmatrix},$$

leading to

$$\mathbf{N} = \mathbf{X}_s \times \mathbf{X}_t = -a\sin t \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a^2\cos s\sin^2 t \\ -a^2\sin s\sin^2 t \\ -a^2\sin t\cos t \end{bmatrix}.$$

This leads to

$$\begin{aligned} ||\mathbf{X}_s \times \mathbf{X}_t|| &= \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2} \\ &= \sqrt{a^4\cos^2 s\sin^4 t + a^4\sin^2 s\sin^4 t + a^4\sin^2 t\cos^2 t} \\ &= \sqrt{a^4\sin^2 t} = a^2\sin t. \end{aligned}$$

where we do not need to shroud this last term in absolute values since $\sin t$ is nonnegative for $t \in [0,\pi]$.

Thus we have

$$\begin{aligned}\mathbf{area}(S_a^2) &= \iint_{S_a^2} dS = \int_0^{\pi}\int_0^{2\pi} ||\mathbf{X}_s \times \mathbf{X}_t||\ ds\,dt \\ &= \int_0^{\pi}\int_0^{2\pi} a^2 \sin t\, ds\, dt = \int_0^{\pi} 2\pi a^2 \sin t\, dt \\ &= -2\pi a^2 \cos t \Big|_0^{\pi} = 2\pi a^2 + 2\pi a^2 = 4\pi a^2.\end{aligned}$$

Example 19.9. ***The surface area of a graph.*** For $f : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}$,

$$\mathcal{S} = \mathbf{X}(\mathcal{D}) = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; z = f(x, y)\right\} = \mathbf{graph}(f).$$

Here

$$\mathbf{X}_x(x, y) = \begin{bmatrix} 1 \\ 0 \\ f_x \end{bmatrix}, \quad \text{and} \quad \mathbf{X}_y(x, y) = \begin{bmatrix} 0 \\ 1 \\ f_y \end{bmatrix},$$

so that

$$\mathbf{X}_x \times \mathbf{X}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k} = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}.$$

Then the surface area of $\mathcal{S} = \mathbf{X}(\mathcal{D}) = \mathbf{graph}(f)$ is

$$\mathbf{area}\,(\mathcal{S}) = \iint_{\mathcal{S}} dS = \iint_{\mathcal{D}} ||\mathbf{X}_s \times \mathbf{X}_t||\ dt = \iint_{\mathcal{D}} \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA.$$

Now compare this to the single variable calculation of the length of a curve which is the graph of a function $f : [a, b] \to \mathbb{R}$:

$$\mathbf{length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$