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LECTURE 18: THE THEOREM OF GREEN.

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Synopsis. Today, we go directly into on of the three big theorem's of vector calculus, Green's Theorem. This theorem exposes a deep relationship between the aggregate behavior of a vector field along the boundary of a relatively nice region in the plane (the vector line integral), to the integral of a related function on the interior of the region. Since Green's Theorem is restricted to regions in the plane, there are a number of ways to craft the related integrals, giving different geometric meaning to the quantities. One interesting geometric interpretation is that the theorem relates the total twisting effect of a vector field in the region (measured by integrating the curl of the vector field as it sits in three space with no vertical component), to the total tangent component of the vector field along the closed boundary. Proving this theorem is neither deep nor long, and we go over the idea here in lecture. We finish with a general definition and discussion of the properties of a special kind of vector field that shows up in an lot of physical applications.

18.1. Green's Theorem.

We begin directly with the theorem:

Theorem 18.1. *Let $\mathcal{D}$ be a closed, bounded region in $\mathbb{R}^2$, whose boundary $\mathbf{c} = \partial\mathcal{D}$ is a finite union of simple, closed, curves, oriented so that $\mathcal{D}$ is always on the left. For a C^1-vector field on $\mathcal{D}$ $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$, we have*

$$\int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s} = \oint_{\mathbf{c}} M\,dx + N\,dy = \iint_{\mathcal{D}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_{\mathcal{D}} (\nabla \times \mathbf{F}) \bullet \mathbf{k}\,dA. \tag{18.1}$$

Some notes:

- The first equal sign is obvious since $\mathbf{F} = \begin{bmatrix} M \\ N \end{bmatrix}$ and $d\mathbf{s} = \begin{bmatrix} dx \\ dy \end{bmatrix}$.
- The last equal sign is also obvious since $(\nabla \times \mathbf{F}) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$. To see this, think of a vector field in $\mathbb{R}^2$ as a vec tor field in $\mathbb{R}^3$ with z-component 0. Then calculate $\nabla \times \mathbf{F}$.
- The theorem basicallt says that the vector line integral (the circulation) of $\mathbf{F}$ along $\partial\mathcal{D}$ equals the curl of $\mathbf{F}$ on $\mathcal{D}$.
  (1) $\int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s}$ measures the aggregate component of $\mathbf{f}$ tangent to $\mathbf{c}$ and in the direction of travel along $\mathbf{c}$.
  (2) Curl in two dimensions measures the rotation or twisting one would experience if flowing along $\mathbf{F}$.

  So the sum total of the push or pull of a particle by $\mathbf{F}$ along $\mathcal{D}$ equals the total rotational effect of $\mathbf{F}$ on $\mathcal{D}$.

Example 18.2. ***A constant vector field.*** For a constant vector field $\mathbf{F}$, we know $\nabla \times \mathbf{F} = \mathbf{0}$. So what can we say about what is happening on the left-hand side of Equation 18.1? How about in the middle?

- The proof is elementary, and relies on three facts:
  (1)

  Lemma 18.3. *If $\mathcal{D}$ is elementary of Type I, then* $\displaystyle\oint_{\partial\mathcal{D}} M\,dx = -\iint_{\mathcal{D}} \frac{\partial M}{\partial y}\,dA.$

  (2)

  Lemma 18.4. *If $\mathcal{D}$ is elementary of Type II, then* $\displaystyle\oint_{\partial\mathcal{D}} N\,dy = \iint_{\mathcal{D}} \frac{\partial N}{\partial x}\,dA.$

  (3) Any region $\mathcal{D}$, valid for Green's Theorem, can be cut up into a finite number of elementary regions, so that (i) the ends of each cut line are in $\partial\mathcal{D}$, (ii) the cuts do not intersect, (iii) $\mathcal{D} = \bigcup \mathcal{D}_i$, with each $\mathcal{D}_i$ elementary of Type III, and (iv) each cut intersects exactly 2 $\mathcal{D}_i$'s with each cut oriented in each $\mathcal{D}_i$ oppositely. Note there, that then, vector line integral along the cut lines will cancel out. So there will be no contribution of the cut lines in the calculation. And within the double integral, the contributions of all cut lines will also be 0.

Here is the idea of the proof of Lemma 18.3:

Proof of Lemma 18.3. With $\mathcal{D}$ elementary of Type I, write

$$\mathcal{D} = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; a \leq x \leq b,\ \alpha(x) \leq y\beta(x) \right\}.$$

Orient $\partial\mathcal{D} = \mathbf{c}_1^+ \cup \mathbf{d}_1^+ \cup \mathbf{c}_2^- \cup \mathbf{d}_2^-$, as needed, where the plus sign means compatible with variable values, and the minus sign means contrary to parameter values. Then, on the right-hand side, we have

$$\begin{aligned} -\iint_{\mathcal{D}} \frac{\partial M}{\partial y}\,dA &= \int_a^b \int_{\alpha(x)}^{\beta(x)} -\frac{\partial M}{|partialy}\,dy\,dx \\ &= -\int_a^b \left(M(x, \beta(x)) - M(x, \alpha(x))\right)\,dx \\ &= \int_a^b \left(M(x, \alpha(x)) - M(x, \beta(x))\right)\,dx \end{aligned}$$

by the Fundamental Theorem of Calculus, suitably adapted. But here

$$-\int_a^b M(x, \beta(x))\,dx = \int_{\mathbf{c}_2^-} M\,dx, \quad \text{and} \quad \int_a^b M(x, \alpha(x))\,dx = \int_{\mathbf{c}_1^+} M\,dx.$$

Now add to this the fact that

$$\int_{\mathbf{d}_1^+} M\,dx = \int_{\mathbf{d}_2^-} M\,dx = 0$$

since x is constant along these curves, and we have

$$\begin{aligned}-\iint_{\mathcal{D}} \frac{\partial M}{\partial y}\, dA &= \int_{\mathbf{c}_1^+} M\, dx + \int_{\mathbf{d}_1^+} M\, dx + \int_{\mathbf{c}_2^-} M\, dx + \int_{\mathbf{d}_2^-} M\, dx \\ &= \oint_{\partial\mathcal{D}} M\, dx.\end{aligned}$$

□

Exercise 1. Prove Lemma 18.4.

18.2. **Conservative Vector Fields.** Here, we have a definition:

Definition 18.5. A C^0-vector field $\mathbf{F}$ has *path-independent line integrals* if for any two piecewise C^1-curves with the same endpoints $\mathbf{x}_1$ and $\mathbf{x}_2$, we have

$$\int_{\mathbf{x}_1} \mathbf{F} \bullet d\mathbf{s} = \int_{\mathbf{x}_2} \mathbf{F} \bullet d\mathbf{s}.$$

Now notice that, since the two curves have the same endpoints, they together form a closed curve (traversing one of them backwards, that is). Sometimes this closed curve is simple.

Theorem 18.6. *A C^0-vector field $\mathbf{F}$ has path-independent line integrals iff $\int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s} = 0$ for every piecewise C^1-simple, closed curve $\mathbf{c}$ in the domain of $\mathbf{F}$.*

Note: If $\mathbf{x}_1^+ \cup \mathbf{x}_2^-$ is not simple, yet consists of a finite number of isolated intersections and/or intervals that coincide, then the result is still true. Why is this the case? Can you reason it through?

Definition 18.7. A C^0-vector field $\mathbf{F}$ is called *conservative* or a *gradient field* if there is a C^1-real valued function f, where $\mathbf{F} = \nabla f$. Such an f is called a *potential* for $\mathbf{F}$, adn is said to *generate* the vector field $\mathbf{F}$.

Notes:

- Conservative vector fields always and only (read: iff) have path-independent line integrals. Indeed,

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{F} \bullet d\mathbf{s} = \int_{\mathbf{x}} \nabla f \bullet d\mathbf{s} &= \int_a^b \nabla f\left(\mathbf{x}(t)\right) \bullet \mathbf{x}'(t)\, dt \\ &= \int_a^b \frac{d}{dt}\left[f\left(\mathbf{x}(t)\right)\right] dt \\ &= f\left(\mathbf{x}(t)\right)\Big|_a^b = f\left(\mathbf{x}(b)\right) - f\left(\mathbf{x}(a)\right),\end{aligned}$$

  depends only on the endpoints of the path. This is Theorem 6.3.3 in the text. And when the curve is closed, the endpoints are the same, so....
- In $\mathbb{R}^2$ and $\mathbb{R}^3$, conservative vector fields are irrotational: If $\mathbf{F}$ is conservative, then $\nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0}$. The converse of this statement, suitably modified, is: If $\mathbf{F}$ is irrotational, AND the domain of $\mathbf{F}$ is *simply connected*, then $\mathbf{F}$ is conservative. This is Theorem 6.3.5 in the text.

Definition 18.8. A region in either $\mathbb{R}^2$ or $\mathbb{R}^3$ is *simply connected* if it is connected (comes in one piece) and every simply closed curve in the region has its entire interior inside the region.

So a disk in the plane in simply connected, but an annulus in the plane is not.

Example 18.9. ***A nonconservative vector field on a non-simply connected domain.*** Let

$$\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + 0\mathbf{k}$$

on $\mathcal{W} = \mathbb{R}^3 - \left\{(0,0,z) \in \mathbb{R}^3 \;\middle|\; z \in \mathbb{R}\right\}$. Here, $\mathbf{F}$ is irrotational ($\nabla \times \mathbf{F} = \mathbf{0}$), but $\mathbf{F}$ is not conservative. It is also the case, though, that $\mathcal{W}$ is not simply connected, as the unit circle in the xy-plane inside $\mathbb{R}^3$ cannot be continuously shrunk to a point in $\mathcal{W}$. I will offer this as an exercise to show that $\mathbf{F}$ is not conservative. As a hint, find a simple, closed curve where the vector line integral of $\mathbf{F}$ along the curve is not 0.

So a good question is: how does one tell if a vector field is conservative? The answers are varied, but include tools like (1) checking the mixed partials of what the potential function would be, (2) integrating to attempt to find the potential, and (3) searching for a closed simple curve on which the vector line integral of the vector field does not vanish.

Example 18.10. ***Find a potential for*** $\mathbf{F} = (y+z)\mathbf{i} + 2z\mathbf{j} + (x+y)\mathbf{k}$***, if conservative.*** The solution here: If a potential f exists, then $\frac{\partial f}{\partial x} = y + z$. But this means that $f(x,y,z) = xy + xz + g(y,z)$. But then

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}(y,z) = 2z.$$

Do you see why this cannot happen? Also,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & 2z & x+y \end{vmatrix} = (1-2)\mathbf{i} + ...$$

and at this point, we can halt the calculation, since the vector field is not irrotational, and hence cannot be conservative.

Example 18.11. ***Find a potential for*** $\mathbf{F} = (2x+y)\mathbf{i} + (z\cos yz + x)\mathbf{j} + (y\cos yz)\mathbf{k}$**.** Here, one first step is to identify that if a function $f : \mathbb{R}^3 \to \mathbb{R}$ exists so that $\nabla f = \mathbf{F}$, then $\frac{\partial f}{\partial x} = 2x + y$. Thus, we can see that

$$f(x,y,z) = x^2 + xy + g(y,z)$$

by anti-differentiation. Thus,

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}(y,z) = x + z\cos yz,$$

so that $\frac{\partial g}{\partial y}(y,z) = z\cos yz$ and $g(y,z) = \sin yz + h(z)$. We now know that

$$f(x,y,z) = x^2 + xy + \sin yz + h(z).$$

Lastly, we have

$$\frac{\partial f}{\partial z} = y\cos yz + h'(z) = y\cos yz$$

so that $h'(z) = 0$, and thus $h(z) = \text{const}$. Thus we can say that

$$f(x, y, z) = x^2 + xy + \sin yz.$$