April 3, 2019

LECTURE 17: THE LINE INTEGRAL.

110.211 HONORS MULTIVARIABLE CALCULUS
PROFESSOR RICHARD BROWN

Synopsis. Here, we dive deeper into integration under the idea that in multiple directions, there are more ways to study the properties of functions. The first type is called the line integral, where one integrates over a curve. The two varieties, the *scalar* and *vector line integrals*, have interesting geometric interpretations as well as simple meaning on their own. Today, we define and study these two types.

Helpful Documents. GIF: `LineIntegralOfScalarField`, `LineIntegralOfVectorField`.

17.1. **Return to notation.** Recall the notation we used when discussing integration:

- In single variable calculus,
$$\int_a^b f(x)\,dx = \int_{\mathcal{I}} f\,dx,$$
where $x \in \mathcal{I} \subset \mathbb{R}$ and f is assumed to be a function of x.
- In vector calculus to date: For a double integral,
$$\iint_{\mathcal{D}} f\,dA, \quad \mathcal{D} \subset \mathbb{R}^2.$$
And for triple integrals,
$$\iiint_{\mathcal{W}} f\,dV, \quad \mathcal{W} \subset \mathbb{R}^3,$$
where f is real-valued, so $f : \mathcal{W} \to \mathbb{R}$.

These are definite integrals whose value would remain unchanged upon a change of variables. Hence the notation denotes a coordinate-free attempt to write the quantities: For $\mathcal{D}$ a region in the xy-plane, for example, then
$$\int_{\mathcal{D}} f\,dA = \iint_{\mathcal{D}} f(x,y)\,dx\,dy.$$
In this lecture, we will define some new ways to study properties of functions over relevant domains in $\mathbb{R}^n$.

17.2. **Line Integrals.**

17.2.1. *Real-valued, scalar functions.* Recall that differentiating a C^1-function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ along a curve $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{R}^n$, where $\mathbf{x}(I) \subset x \subset \mathbb{R}^n$, looks a lot like a single variable calculus endeavor:

$$\begin{aligned}\frac{df}{dt}(t) &= \frac{d}{dt} f\left(\mathbf{x}(t)\right) = Df\left(\mathbf{x}(t)\right) \cdot \frac{d\mathbf{x}}{dt}(t) \\ &= \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} \in \mathbb{R}.\end{aligned}$$

One would expect, then, that integrating a funciton $f : X \subset \mathbb{R}^n \to \mathbb{R}$ over a curve (in essence, adding up its values along the curve) should also seem like a 1-dimenaional calculation. In fact, it is quite similar, once we can interpret things correctly:

Definition 17.1. Given an integrable $f : X \subset \mathbb{R}^n \to \mathbb{R}$, and a C^1-curve $\mathbf{x} : \mathcal{I} \to \mathbb{R}^n$, where $\mathcal{I} = [a, b] \subset \mathbb{R}$ and $\mathbf{x}(\mathcal{I}) \subset X$, the *scalar line integral* of f over $\mathbf{x}$ is

$$\int_{\mathbf{x}} f\, ds = \int_a^b f\left(\mathbf{x}(t)\right) ||\mathbf{x}'(t)||\, dt.$$

Some notes:

- The symbolic s denotes arclength: Recall that, for any curve parameterization $\mathbf{x} : [a, b] \to \mathbb{R}$, the arclength parameter is

  $$s(t) = \int_a^t ||\mathbf{x}'(\tau)||\, d\tau.$$

  Seen as a change in variables, the new differential is then, by the Fundamental Theorem of Calculus

  $$ds = s'(t)\, dt = \left(\frac{d}{dt}\left[\int_a^t ||\mathbf{x}'(\tau)||\, d\tau\right]\right) dt = ||\mathbf{x}'(t)||\, dt.$$

  This suggests that the (scalar) line integral is *parameterization independent*. This suggestion is correct.
- The scalar line integral is also sometimes called the *path integral*, or the *line integral of a scalar field*.
- Here is a good geometric interpretation of the scalar line integral, using a function on two variables so that we can see the graph, $f : X \subset \mathbb{R}^2 \to \mathbb{R}$: The $\mathbf{graph}(f)$, as a subset of $\mathbb{R}^3$ defined as the set of solutions to $z = f(x, y)$, sits vertically over X sitting within the xy-plane in $\mathbb{R}^3$. The curve $\mathbf{x}$ sits inside X. Form a vertical wall by drawing a vertical line from each point in $\mathbf{x} \subset X$ to $\mathbf{graph}(\mathbf{x}) \subset \mathbf{graph}(f)$. Then the scalar line integral of f along $\mathbf{x}$ is the total area of this wall. This is simply the curved version of the standard geometric interpretation of the integral of a function $f(x)$ over an interval in Calculus I. See Figure **??**.
- So, again, the scalar line integral of f along $\mathbf{x}$ is essentially a 1-dimensional calculation, but now in $\mathbb{R}^n$, $n > 1$.

Example 17.2. ***A constant function along a straight line.*** Let $f(x,y)=4$, and $\mathbf{x}(t)=\begin{bmatrix}3\\4\end{bmatrix}t$, defined on $\mathcal{I}=[0,1]$. Then

$$\begin{aligned}\int_{\mathbf{x}} f\,ds &= \int_0^1 f\left(\mathbf{x}(t)\right)||\mathbf{x}'(t)||\,dt\\ &= \int_0^1 f\left(3t,4t\right)\sqrt{3^2+4^2}\,dt\\ &= \int_0^1 4\sqrt{25}\,dt = 20t\Big|_0^1 = 20.\end{aligned}$$

Perhaps this was rather obvious, since the rectangle (in $\mathbb{R}^3$), whose base if $\mathbf{x}([0,1])$, and whose height is 4 has area which is the length of the curve times the height: The length of the curve is 5, so the area is $4\cdot 5=20$.

Example 17.3. ***Surface area of a cylinder.*** The surface area SA of a cylinder of radius r and height h is $SA=2\pi rh$ (there is no top nor bottom to this cylinder). We can functionally calculate this: Let $g(x,y)=h$, and define $\mathbf{c}(t)=\begin{bmatrix}r\cos t\\ r\sin t\end{bmatrix}t$ be a curve, defined on $\mathcal{I}=[0,2\pi]$. Then

$$\int_{\mathbf{c}} g\,ds = \int_0^{2\pi} h\sqrt{r^2\cos^2 t+r^2\sin^2 t}\,dt = \int_0^{2\pi} hr\,dt = 2\pi rh.$$

Example 17.4. ***A curve in*** $\mathbb{R}^3$***.*** Integrate $f(x,y,z)=\frac{xz}{y}$ over $\mathbf{x}(t)=\begin{bmatrix}4\\2t\\\frac{t^2}{2}\end{bmatrix}$, defined on $\mathcal{I}=[1,2]$. First, we calculate

$$||\mathbf{x}'(t)||=\sqrt{0^2+2^2+t^2}=\sqrt{4+t^2}.$$

Then we can calculate the integral:

$$\int_{\mathbf{x}} f\,ds = \int_1^2 \frac{(4)\left(\frac{t^2}{2}\right)}{2t}\sqrt{4+t^2}\,dt = \int_1^2 2t\sqrt{4+t^2}\,dt.$$

With the substitution $u=4+t^2$, along with $du=2t\,dt$, we get

$$\int_1^2 2t\sqrt{4+t^2}\,dt = \int_5^8 \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_5^8 = \frac{2}{3}\left(16\sqrt{2}-5\sqrt{5}\right).$$

17.2.2. *Real-valued, vector functions (vector fields).* For a curve $\mathbf{x}:\mathcal{I}\to\mathbb{R}^n$, $\mathcal{I}=[a,b]\subset\mathbb{R}$, where $\mathbf{x}(I)\subset x\subset\mathbb{R}^n$, inside a vector field $\mathbf{F}:X\subset\mathbb{R}^n\to\mathbb{R}^n$ (it is again understood that the entire curve is in X), one can ask how much of the vector field can be "seen" by (a point on) the curve. Recall for a curve where $\mathbf{x}'(t)\neq 0$, $\forall t\in\mathcal{I}$, that $\mathbf{T}(t)=\frac{\mathbf{x}'(t)}{||\mathbf{x}'(t)||}$ is the unit tangent vector at t. Then the component of $\mathbf{F}$ along $\mathbf{x}$ is

$$\mathbf{F}\left(\mathbf{x}(t)\right)\bullet\mathbf{T}(t).$$

We can, again, "add up" these values all along the curve to get

$$\int_{\mathbf{x}} (\mathbf{F} \bullet \mathbf{T}) \, ds,$$

the scalar line integral of $\mathbf{F} \bullet \mathbf{T}$, as a scalar-valued function, along $\mathbf{x}$.

Note, that, in a way, this represents the aggregate boost or hindrance a particle moving along the curve would feel by the vector field.

But

$$\begin{aligned}
\int_{\mathbf{x}} (\mathbf{F} \bullet \mathbf{T}) \, ds &= \int_a^b \left(\mathbf{F}(\mathbf{x}(t)) \bullet \frac{\mathbf{x}'(t)}{||\mathbf{x}'(t)||} \right) ||\mathbf{x}'(t)|| \, dt \\
&= \int_a^b \mathbf{F}(\mathbf{x}(t)) \bullet \mathbf{x}'(t) \, dt \\
&= \int_a^b \mathbf{F}(\mathbf{x}(t)) \bullet \, d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \bullet \, d\mathbf{s},
\end{aligned}$$

where

$$d\mathbf{s} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = \begin{bmatrix} x_1'(t)\, dt \\ \vdots \\ x_n'(t)\, dt \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} dt = \mathbf{x}'(t)\, dt$$

is the vector infinitesimal displacement and represents an infinitesimal change in displacement along each coordinate direction (instead of just along the curve).

Definition 17.5. For a C^1-curve $\mathbf{x} : [a, b] \to X \subset \mathbb{R}^n$ in a vector field $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$, the *vector line integral* of $\mathbf{F}$ along $\mathbf{x}$ is

$$\int_{\mathbf{x}} \mathbf{F} \bullet \, d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \bullet \mathbf{x}'(t) \, dt.$$

Some notes:

(1) Recall that the work W done by a vector field $\mathbf{F}$ (force field) on a particle is just the force times the displacement $\mathbf{d}$. As vector quantities, when the particle's motion is
- linear:

  $W = \mathbf{F} \bullet \mathbf{d} = ||\mathbf{F}|| \, ||\mathbf{d}|| \cos\theta = ||\mathbf{F}|| \cdot (\text{displacement in direction of } \mathbf{F})$.
- curved: Here we must measure infinitesimally, and

  $$W = \int_a^b \mathbf{F}(\mathbf{x}(t)) \bullet \mathbf{x}'(t) \, dt,$$

  a scalar integral.

(2) Some facts:

Theorem 17.6. *A scalar line integral is independent of the curve's parameterization.*

Theorem 17.7. *A vector line integral depends on a curve's parameterization only in the direction of travel.*

(3) Some curve facts:

- For $\mathcal{I} = [a, b]$, and $\mathbf{x} : \mathcal{I} \to \mathbb{R}^n$, a piecewise C^1-continuous curve, a C^1-function $h : \mathcal{J} = [c, d] \to \mathcal{I}$, which is 1-1, onto, and has a C^1-inverse, induces $\mathbf{p} : \mathcal{J} \to \mathbb{R}^n$, $\mathbf{p} = \mathbf{x} \circ h$ a *reparameterization.*
- For $\mathbf{x}$ 1-1 on $\mathcal{I}$, then $\mathbf{x}(\mathcal{I})$ has only two directions of travel. A choice of direction, given by the notion of increasing values of t along $\mathbf{x}$, is called an *orientation* of the curve.
- A reparameterization is called *oriantation preserving* if the direction of travel along $\mathbf{p}(\mathcal{J})$ is the same as that along $\mathbf{x}(\mathcal{I})$. Otherwise, the reparameterization is called *orientation reversing.*
- Again, for $\mathcal{I} = [a, b]$, a curve $\mathbf{x} : \mathcal{I} \to \mathbb{R}^n$ is called *simple* if $\mathbf{x}$ is 1-1 on (a, b), and *closed* is $\mathbf{x}(a) = \mathbf{x}(b)$.
- In general, scalar line (path) integrals are defined on curves, and vector line integrals are defined on oriented curves.

(4) If $\mathbf{x}$ is a *simple, closed curve*, then the notation for a vector line integral is

$$\oint_{\mathbf{x}} \mathbf{F} \bullet d\mathbf{s}$$

and is called the *circulation* of $\mathbf{F}$ along $\mathbf{x}$. We will see the geometric interpretation of this in time.

(5) For a vector line integral, the term

$$\mathbf{F} \bullet d\mathbf{s} = F_1\, dx_1 + \ldots + F_n\, dx_n = \sum_{i=1}^{n} F_i\, dx_i$$

is called a *differential 1-form.* We will devote more time to this later.

Example 17.8. ***Evaluate the integral***

$$\oint_{\mathbf{c}} (x^2 - y^2)\, dx + (x^2 + y^2)\, dy,$$

where c is the boundary of the unit square, oriented clockwise. Recognizing that this integral is simply a vector line integral of the vector field $\mathbf{F} = (x^2 - y^2)\mathbf{i} + (x^2 + y^2)\mathbf{j}$ over the closed, simple curve $\mathbf{c}$ given by the edge of the unit square, one sees that

$$(x^2 - y^2)\, dx + (x^2 + y^2)\, dy = \mathbf{F} \bullet d\mathbf{s}$$

is just a differentiable 1-form. The process here would be, then, the parameterize the unit square perimeter by time, and integrate under the parameterization: We get

$$\mathbf{c}(t) = \begin{cases} (0, t) & 0 \le t \le 1 \\ (t-1, 1) & 1 \le t \le 2 \\ (1, 3-t) & 2 \le t \le 3 \\ (4-t, 0) & 3 \le t \le 4 \end{cases}$$

as our clockwise parameterization, beginning and ending at the origin. To understand the switch to the parameterization, we highlight the first "piece": Along the left-side edge of the unit square only, the parameterization is the path $\mathbf{c}_1$, going from $(0, 0)$ to $(0, 1)$ and parameterized by t in the y-direction only. We get

$$\begin{aligned}\int_{\mathbf{c}_1} \mathbf{F} \bullet d\mathbf{s} &= \int_{\mathbf{c}_1} (x^2 - y^2)\, dx + (x^2 + y^2)\, dy \\ &= \int_0^1 F_1(x(t), y(t))x'(t)\, dt + F_2(x(t), y(t))y'(t)\, dt \\ &= \int_0^1 \left((0)^2 - (t)^2\right)(0\, dt) + \left((0)^2 + (t)^2\right)(1\, dt) \\ &= \int_0^1 t^2\, dt = \left.\frac{t^3}{3}\right|_0^1 = \frac{1}{3}.\end{aligned}$$

Hence on the four pieces (so once around the square), we get

$$\begin{aligned}\oint_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s} &= \oint_{\mathbf{c}} (x^2 - y^2)\, dx + (x^2 + y^2)\, dy \\ &= \int_0^1 t^2\, dt + \int_1^2 \left((t-1)^2 - 1^2\right)\, dt + \int_2^3 \left(1^2 + (3-t)^2\right)\, dt + \int_3^4 (4-t)^2\, dt \\ &= \int_0^1 t^2\, dt + \int_1^2 \left(t^2 - 2t\right)\, dt + \int_2^3 \left(10 - 6t + t^2\right)\, dt + \int_3^4 \left(16 - 8t + t^2\right)\, dt \\ &= \frac{1}{3} + \left.\left(\frac{t^3}{3} - t^2\right)\right|_1^2 + \left.\left(10t - 3t^2 + \frac{t^3}{3}\right)\right|_2^3 + \left.\left(16t - 4t^2 + \frac{t^3}{3}\right)\right|_3^4 \\ &= \frac{1}{3} + \left(\frac{8}{3} - 4 - \frac{1}{3} + 1\right) + \left(30 - 27 + 9 - 20 + 12 - \frac{8}{3}\right) + \left(64 - 64 + \frac{64}{3} - 48 + 36 - 9\right) \\ &= \frac{1}{3} - \frac{2}{3} + \frac{4}{3} + \frac{1}{3} = \frac{4}{3}.\end{aligned}$$

Example 17.9. ***Calculate the circulation of*** $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ ***over the unit circle in the plane.*** Here, we can parameterize the unit circle as the simple closed curve $\mathbf{x}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, for t on the interval $t \in [0, 2\pi]$. So

$$\begin{aligned}\oint_{\mathbf{x}} \mathbf{F} \bullet d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}\left(\mathbf{x}(t)\right) \bullet \mathbf{x}'(t)\, dt \\ &= \int_0^{2\pi} \left(\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \bullet \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \right) dt = \int_0^{2\pi} 0\, dt = 0.\end{aligned}$$

Note that, here, the vector field is perpendicular to the curve everywhere. Hence there is no circulation of $\mathbf{F}$ along $\mathbf{x}$ in this case.

Example 17.10. ***Circulation of a vector field along one if its integral curves.*** By definition, if $\mathbf{x}$ is an integral curve of a vector field $\mathbf{F}$ on some interval $[a, b]$, then for all $t \in [a, b]$, we have $\mathbf{x}'(t) = \mathbf{F}\left(\mathbf{x}(t)\right)$. So

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{F} \bullet d\mathbf{s} &= \int_a^b \mathbf{F}\left(\mathbf{x}(t)\right) \bullet \bullet \mathbf{x}'(t)\, dt \\ &= \int_a^b \mathbf{x}'(t) \bullet \mathbf{x}'(t)\, dt = \int_a^b \left|\left|\mathbf{x}'(t)\right|\right|^2\, dt.\end{aligned}$$

In this case, we are again simply adding up the vector field components along the curve, but here they equal the velocity vectors all along the curve.