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LECTURE 15: THE DEFINITE TRIPLE INTEGRAL.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. Today we continue the general idea of integration of a real-valued function on more than one variable by generalizing the 2-dimensional version to three dimensions. There is little that is new here except for the pattern of the generalization that leads to the n-dimensional version. Fubini's Theorem still holds, and switching the order of integration outside of a cuboid region still involves checking that the region is elementary in different permutations of the variables of integration and that, if so, one can rewrite the limits as functions of some of the variable properly.

Volumes of higher dimensional regions. Let

$$\mathcal{B} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} a \leq x \leq b \\ c \leq y \leq d \\ p \leq z \leq q \end{array} \right\}$$

be a *cuboid*. We can approximate the volume of the four-dimensional solid $\mathcal{S}$ with $\mathcal{B}$ as its base (can you envision this?) and $\mathbf{graph}(f)$ as its *roof*, where $f : \mathcal{B} \to \mathbb{R}$ is a nonnegative C^0-function, by:

- Partitioning all three dimensions of $\mathcal{B}$ so that
$$\mathcal{B}ijk = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} x \in [x_{i-1}, x_i] \\ y \in [y_{j-1}, y_j] \\ z \in [z_{k-1}, z_k] \end{array} \right\},$$
with lengths $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, and $\Delta z_k = z_k - z_{k-1}$. Note that, in this construction, we will choose a partition size of n for all three dimensions. This will greatly simplify the construction;
- Find $\mathbf{volume}(\mathcal{B}_{ijk}) = \Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$;
- Choose a point $\mathbf{c}_{ijk} \in \mathcal{B}_{ijk}$, $\forall i,j,k \in \{1, \ldots, n\}$;
- Sum over indices,
$$\mathbf{volume}(\mathcal{S}) \approx \sum_{i,j,k=1}^{n} f(\mathbf{c}_{ijk}) \Delta V_{ijk}.$$

Then the triple integral of f over $\mathcal{B}$ is

$$\mathbf{volume}(\mathcal{S}) = \iiint_{\mathcal{B}} f\, dV = \lim_{n\to\infty} \sum_{i,j,k=1}^{n} f(\mathbf{c}_{ijk}) \Delta V_{ijk} = \lim_{n\to\infty} \sum_{i,j,k=1}^{n} f(x_i, y_j, z_k) \Delta x_i \Delta y_j \Delta z_k.$$

Here are some facts about triple integrals:

(1) Fubini's Theorem still holds in higher dimensions.

(2) A region $\mathcal{W} \in \mathbb{R}^3$ is called *elementary* if it can be written as

$$\mathcal{W} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} a \leq x \leq b \\ \alpha(x) \leq y \leq \beta(x) \\ \varphi(x,y) \leq z \leq \psi(x,y) \end{array} \right\},$$

or some permutation of these variables. Here, one direction should look like an interval, a second direction should look like the difference between two functions of that first variable, and the third direction should look like the difference between two functions of the other two.

(3) Given an elementary $\mathcal{W}$, then, if f is continuous on $\mathcal{W}$, we have

$$\iiint_{\mathcal{W}} f\, dV = \int_a^b \int_{\alpha(x)}^{\beta(x)} \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\, dz\, dy\, dx,$$

or, again, some permutation of the three variables. Note again that the integrals are nested here.

(4) The volume of the solid $\mathcal{W}$ can be found by integrating the *unit function* $f(x,y,z) = 1$, so

$$\textbf{volume}(\mathcal{W}) = \iiint_{\mathcal{W}} 1\, dV = \iiint_{\mathcal{W}} dV.$$

(5) Sometimes, it is advantageous to understand that

$$\iiint_{\mathcal{W}} f\, dV = \iint_{\mathcal{D}} \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\, dz\, dA,$$

for $\mathcal{D}$ elementary in x and y, and $\mathcal{W}$ elementary in all three variables.

Example 15.1. ***What is the volume of the unit sphere in*** $\mathbb{R}^3$***?*** Here we define the unit 2-sphere as

$$S^2 = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 = 1 \right\},$$

noting that the notation is common in mathematics and generalizes to S^n, the unit n-sphere, $n \in \mathbb{N}$, as a subset of $\mathbb{R}^{n+1}$. One ways to think of this is the set of all unit-length vectors in $(n+1)$-space. So what does S^1 look like? How about S^0??

Here, then, the space consisting of S^2 and its interior is sometimes called the (unit) 3-ball, or B^3. So we are looking for the volume of B^3. Note that B^3 is elementary, and can be written as the difference between two functions $z = -\sqrt{1-x^2-y^2}$ and $z = \sqrt{1-x^2-y^2}$, on the domain

$$\mathcal{D} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1 \right\} = \left\{ (x,y,z) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1,\ z = 0 \right\}.$$

These graphs of these two functions are the southern and northern hemispheres of S^2, respectively, and meet at the equator.

So here,

$$\textbf{volume}(B^3) = \iiint_B i\, dV = \iint_{\mathcal{D}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx.$$

Example 15.2. ***Let $\mathcal{W}$ be the region bounded by $y = x^2$ and $y + z = 9$ and the xy-plane. Integrate $f(x, y, z) = 8xyz$ over $\mathcal{W}$.*** Here, the roof of this solid is the inclined plane $z = 9 - y$, the floor is the xy-plane, and the wall is given by the parabolic $y = x^2$, projected vertically out of the floor. We get Figure 15.1 below. $\mathcal{W}$ is an elementary region, and one way to see this is the following: As z goes from the floor at 0 to the roof at $9 - y$, the variable y goes from 9 to x^2, and x ranges from -3 to 3. Hence the integration is

$$\begin{aligned}
\int_{-3}^{3}\int_{x^2}^{9}\int_{0}^{9-y} 8xyz\,dz\,dy\,dx &= \int_{-3}^{3}\int_{x^2}^{9}\left[8xy\left(\frac{z^2}{2}\right)\Bigg|_{0}^{9-y}\right]dy\,dz \\
&= \int_{-3}^{3}\int_{x^2}^{9} 4xy(9-y)^2\,dy\,dx \\
&= \int_{-3}^{3}\int_{x^2}^{9} 4x\left(81y - 18y^2 + y^3\right)\,dy\,dx \\
&= \int_{-3}^{3} 4x\left[\left(\frac{81}{2}y^2 - \frac{18}{3}y^3 + \frac{y^4}{4}\right)\Bigg|_{x^2}^{9}\right]dx \\
&= \int_{-3}^{3} 4x\left[\frac{81^2}{2} - 6(9)^3 + \frac{9^4}{4} - \frac{81}{2}x^4 + 6x^6 - \frac{x^8}{4}\right]dx = 0.
\end{aligned}$$

A good question to ask is: Why is this quantity 0? One can "see" that this is true at this point due to the properties of the integral one learned in single variable calculus. Indeed, notice that the integrand is actually an *odd function*, symmetric with respect to the origin. In this case, the integrans is a polynomial with all of the monomials of odd degree, and that the interval one is integrating over is of the form $[-a, a]$, for some $a \geq 0$. Hence one can cease calculating here and conclude.

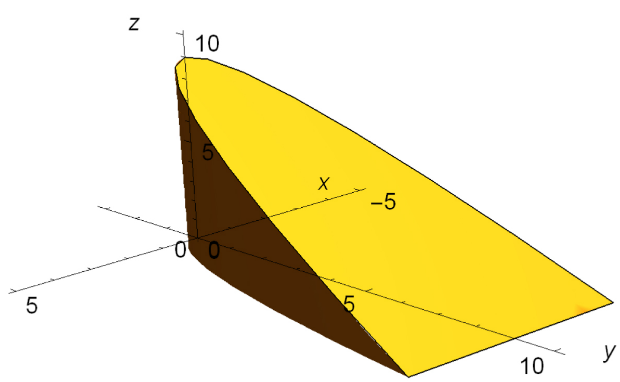

FIGURE 15.1. The solid $\mathcal{W}$ in Example 15.2.