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LECTURE 14: THE DEFINITE INTEGRAL.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. The integral calculus of functions of more than one variable also follows closely the structure and patterns of single variable calculus. However, noting that graphs of functions, even of two independent variables are no longer curves, but hypersurfaces in R^{n+1}, the idea of "area under a curve" must be suitably generalized. In this lecture, we lay the groundwork to understand volumes in many dimensions and what is means to calculate. Then we alter the idea of single variable integration to fit this new multidimensional arena and build the tools and structures we need to create the integral calculus.

Volumes of regions. The area of a two dimensional region $\mathcal{R}$ (say the difference between the graphs of two function over the same domain in single variable calculus, is really just a "sum" of the lengths of all of the vertical lines formed by slicing (along lines of constant values of the independent variable x) on some interval of x comprising the region. Using Figure **??** below, we have

$$\text{Area}(\mathcal{R}) = \int_a^b \ell(x)\, dx,$$

where for each value of $x \in [a, b]$, the value of $\ell(x)$ is $f(x)-g(x)$ on the left and $f(x)-0 = f(x)$ on the right.

This remains true in higher dimensions, at least once we understand the notions of lengths and areas in higher dimensions: Given $f(x, y)$, a nonnegative function defined and continuous on the rectangle

$$\mathcal{R} = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; a \leq x \leq b,\ c \leq y \leq d\right\}, \tag{14.1}$$

its graph lies "over" the region in the xy-plane in $\mathbb{R}^3$ as the set of points $(x, y, 0) \in \mathbb{R}^3$, where $(x, y) \in \mathcal{R}$. Add in the vertical walls connecting the four edges of $\mathcal{R}$ in the floor to the corresponding graphs of the edges in **graph**(f), and one obtains a solid region in $\mathbb{R}^3$, as in Figure **??**, which we define as

$$\mathcal{S} = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; a \leq x \leq b,\ c \leq y \leq d,\ 0 \leq z \leq f(x, y)\right\}.$$

We can calculate the volume of $\mathcal{S}$ as the sum of all of the areas of the "vertical" slices through $\mathcal{S}$ by, say, slicing along lines of constant x, which we will call $\mathcal{R}_x$. In this way, we could write

$$\text{Volume}(\mathcal{S}) = \int_a^b \text{Area}(\mathcal{R}_x)\, dx.$$

Note that we could also slice vertically along lines of constant y, creating regions $\mathcal{R}_y$, so that

$$\text{Volume}(\mathcal{S}) = \int_c^d \text{Area}(\mathcal{R}_y)\; dy.$$

We will stick with the former for now.

So, for each value of x, what is the area of each $\mathcal{R}_x$? At a point $x_0 \in [a,b]$, the area of $\mathcal{R}_{x_0}$ is

$$\text{Area}(\mathcal{R}_{x_0}) = \int_c^d \ell(y)\, dy = \int_c^d f(x_0, y)\, dy.$$

Nesting these two concepts together, we arrive at

$$\text{Volume}(\mathcal{S}) = \int_a^b \text{Area}(\mathcal{R}_x)\, dx = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx = \int_a^b \int_c^d f(x,y)\, dy\, dx.$$

Some notes:

- The parentheses distinguishing the "inside" integral from the "outside" (in the penultimate expression) are not strictly needed (and hence removed) if one understands that the integrals are always taken to be nested.
- The use of the choice of $x = x_0$ subscript is also not needed, and hence removed. It is understood here that as one integrates with respect to one variable, the other is considered fixed, like a parameter. Do you recall this idea from the notion of partial differentiation?
- If the limits of the variables do not depend on each other, then the region one is integrating over is rectangular. In this case, one can reverse the process and form a nested pair of integrals with the order of integration reversed, but with the same result. So

$$\text{Volume}(\mathcal{S}) = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

The general notion of parameterizing the parallel slices (in any chosen direction) through a solid to find its volume is known as *Cavalieri's Principle*. Let $\mathcal{S} \subset \mathbb{R}^n$ be an n-dimensional solid in n-space, bounded in the x_1-direction by $[a,b]$. Then

$$\text{Volume}(\mathcal{S}) = \int_a^b \text{Volume}(\mathcal{R}_{x_1})\, dx_1,$$

where $\text{Volume}(\mathcal{R}_{x_1})$ is the volume of the $(n-1)$-dimensional x_1-slice through $\mathcal{S}$ at $x_1 \in [a,b]$. Recursively speaking, calculating the volume of $\mathcal{S}$ will involve a nested set of n integrals, or an *n-tuple integral*. Note that, while we would easily use terms like *quadruple integral* or *quintuple integral* for volumes in, respectively, $\mathbb{R}^4$ and $\mathbb{R}^5$, we commonly refer to a nested set of three integrals as a *triple integral*, and in two dimensions, a *double integral*.

Now one can define a double integral on a rectangular region $\mathcal{R}$ via a 2-dimensional Riemann Sum:

Define a nonnegative $f(x,y)$ on the region $\mathcal{R}$ defined above in Equation 14.1, and partition $\mathcal{R}$ into boxes by partitioning the two intervals $[a,b]$ and $[c,d]$:

$$\begin{aligned} a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \quad &\text{and} \\ c = y_0 < y_1 < \cdots < y_{m-1} < y_m = d, \quad & \end{aligned}$$

so that $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$. Then the area of the ijth box is then $\Delta A_{ij} = \Delta x_i \Delta y_j$.

Now choose a point (p_i, q_j) within each box, where $p_i \in [x_{i-1}, x_i]$ and $q_j \in [y_{j-1}, y_j]$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then we can approximate the volume of the solid $\mathcal{S}$ between the floor (the domain $\mathcal{R}$ in the xy-plane in $\mathbb{R}^3$, the ceiling (the **graph**(f) over $\mathcal{R}$, by the sum of all of the volumes of the small cuboids whose base in $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$, height is $f(p_i, q_j)$. Hence

$$\text{Volume}(\mathcal{S}) \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i, q_j) \Delta A_{ij}.$$

This is a 2-dimensional Riemann Sum.

Definition 14.1. The double integral of f on $\mathcal{R}$ is

$$\iint_{\mathcal{R}} f \, dA = \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i, q_j) \Delta A_{ij}$$

when the limits exists.

Notes:

(1) Actually, as stated, the definition has a serious flaw in it. I will leave it unspecified to see if you can see it. It is a flaw in the nature of the limit. Find it!
(2) If the limit exists, then we say f is *integrable* on $\mathcal{R}$.
(3) The notation used, without specific upper and lower limits but the more general $\mathcal{R}$ under the double integral sign, is common and accentuates the region $\mathcal{R}$ instead of the coordinates used. But, using the standard cartesian coordinates x, and y, we automatically know then, that, in this case,

$$\iint_{\mathcal{R}} f \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.$$

(4) Over any base box in a Riemann Sum, if $f(x, y) < 0$, then we interpret the volume of that box as negative, just like in single variable calculus.
(5) Also like in single variable calculus, the same problems and caveats that occur with the limit can occur here also:
   - In single variable calculus, piecewise continuous functions on an interval $[a, b]$ are integrable. Recall that piecewise continuous functions are those that are continuous everywhere, except on a finite set of points where "jump" discontinuities can occur.
   - In two dimensions, if f is bounded on $\mathcal{R}$ with the set of all discontinuities having zero area, then f is integrable. One way to see this is to think of **graph**(f) as smooth but possibly cut up into a finite number of pieces.
   - Continuous functions on closed, bounded domains are always integrable.

Theorem 14.2 (Fubini)**.** *Let f be bounded on $\mathcal{R} = [a, b] \times [c, d]$ and assume that the set S of discontinuities of f on $\mathcal{R}$ has zero area. If every line parallel to the coordinate axes meets S in, at most, a finite number of places, then*

$$\iint_{\mathcal{R}} f \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.$$

Notes:

- The fact that $\mathcal{R}$ is a rectangle is vital here.

- The stipulation that all lines parallel to coordinate axes meet S in at most a finite number of places is sufficient but not strictly necessary. It forces the function intersecting the slice to be piecewise continuous, and thus integrable. But this is not the only way to have an integrable funciton on each slice.
- For all intents and purposes, aero area means that the set of discontinuities has smaller dimension as a set than $\mathcal{R}$.

The properties of double integrals reflect those of their 1-dimensional cousins. See Proposition 5.2.7 on page 320 of the text.

Keep in mind that Cavalieri's Principle will still hold for solids in 3-space defined for regions of the plane (as domains for functions) more general than rectangles. However, the order of integration, when defining and calculating a double integral, may matter. Hence, we need to understand why and how.

Definition 14.3. A region $\mathcal{D} \subset \mathbb{R}^2$ is called *elementary* if it can be described via an interval in one variable and as the difference between two functions of that variable in the other. There are three types:

(1) **Type I:** $\mathcal{D} = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, a \leq x \leq b, \gamma(x) \leq y \leq \delta(x)\right\}$, where $\gamma(x)$ and $\delta(x)$ are continuous functions on $[a,b]$.
(2) **Type II:** $\mathcal{D} = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, c \leq y \leq d, \alpha(x) \leq x \leq \beta(x)\right\}$, where $\alpha(y)$ and $\beta(y)$ are continuous functions on $[c,d]$.
(3) **Type III:** $\mathcal{D}$ is of both Type I and Type II.

A region $\mathcal{D}$ is called *non-elementary* if it is neither Type I nor Type II. We immediately have:

Theorem 14.4. *If $\mathcal{D} \subset \mathbb{R}^2$ is elementary and f is C^0 on $\mathcal{D}$, then*

(1) **Type I:** $\displaystyle\iint_{\mathcal{D}} f\, dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y)\, dy\, dx,$

(2) **Type II:** $\displaystyle\iint_{\mathcal{D}} f\, dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)\, dx\, dy.$

Note here that the proof is noneventful, and relies on a notion of extending $\mathcal{D}$ to some rectangular region $\mathcal{R} \supset \mathcal{D}$ by creating a new function f^{ext} on $\mathcal{R}$ which equals f on $\mathcal{D}$ and is 0 outside of $\mathcal{D}$ in $\mathcal{R}$. This creates a discontinuous function on $\mathcal{R}$, but one that is integrable according to Fubini. Do not worry about this technique. It works for the theorem, but is not necessary to know for a good understanding of integration.

Example 14.5. Let $f(x) = -\frac{3}{2}x + 4$, and $g(x) = \frac{1}{2}x$. The region $\mathcal{D}$ in the (closed) first quadrant of the plane between these two functions is a triangle of height 2, with base along the y-axis from 0 to 4. Integrate the function $h(x,y) = 2x + 2y$ on $\mathcal{D}$.

Strategy: View $\mathcal{D}$ as elementary of either type and construct the double integral according to Theorem 14.4. Then use the Fundamental Theorem of Calculus (from single variable calculus) on the "inside" integral, then again on the "outside" integral.

Solution: Viewing $\mathcal{D}$ as a Type I elementary region, we set $\gamma(x) = g(x)$ and $\delta(x) = f(x)$, and use the formula of Theorem 14.4 to set up the integral. We get

$$\iint_{\mathcal{D}} f\ dA = \int_0^2 \int_{\frac{1}{2}x}^{-\frac{3}{2}x+4} (2x+2y)\ dy\, dx.$$

Then we calculate:

$$\begin{aligned}
\iint_{\mathcal{D}} f\ dA &= \int_0^2 \int_{\frac{1}{2}x}^{-\frac{3}{2}x+4} (2x+2y)\ dy\, dx = \int_0^2 \left[\left(2xy + y^2\right) \Big|_{\frac{1}{2}x}^{-\frac{3}{2}x+4} \right] dx \\
&= \int_0^2 \left[2x\left(-\frac{3}{2}x+4\right) + \left(-\frac{3}{2}x+4\right)^2 - \left(2x\left(\frac{1}{2}x\right) + \left(\frac{1}{2}x\right)^2\right) \right] dx \\
&= \int_0^2 \left[-3x^2 + 8x + \frac{9}{4}x^2 - 12x + 16 - x^2 - \frac{1}{4}x^2 \right] dx \\
&= \int_0^2 \left(-2x^2 - 4x + 16\right)\ dx \\
&= \left(-\frac{2}{3}x^3 - 2x^2 + 16x\right) \Big|_0^2 = -\frac{16}{3} - 8 + 32 = \frac{56}{3}.
\end{aligned}$$

Notice that we can also deem $\mathcal{D}$ as elementary of Type II, using

$$\alpha(y) = 0, \quad \text{and} \quad \beta(y) = \begin{cases} -\frac{2}{3}(y-4) & y \in [1,4] \\ 2y & y \in [0,1]. \end{cases}$$

Then the construction and calculation become

$$\begin{aligned}
\iint_{\mathcal{D}} f\, dA &= \int_0^4 \int_{\alpha(y)}^{\beta(y)} (2x+2y)\, dx\, dy \\
&= \int_0^1 \int_0^{2y} (2x+2y)\, dx\, dy + \int_1^4 \int_0^{-\frac{2}{3}(y-4)} (2x+2y)\, dx\, dy \\
&= \int_0^1 \left[\left(x^2+2xy\right)\Big|_0^{2y} \right] dy + \int_1^4 \left[\left(x^2+2xy\right)\Big|_0^{-\frac{2}{3}(y-4)} \right] dy \\
&= \int_0^1 \left((2y)^2 + 2(2y)y\right) dy + \int_1^4 \left(\left(-\frac{2}{3}(y-4)\right)^2 + 2\left(-\frac{2}{3}(y-4)\right) y \right) dy \\
&= \int_0^1 8y^2\, dy - \frac{2}{3}\int_1^4 \left(-\frac{2}{3}(y^2-8y+16) + 2y^2 - 8y \right) dy \\
&= \left[\frac{8}{3}y^3 \Big|_0^1 \right] - \frac{2}{3}\int_1^4 \left(\frac{4}{3}y^2 - \frac{8}{3}y - \frac{32}{3} \right) dy \\
&= \frac{8}{3} - \frac{8}{9}\left[\left(\frac{y^3}{3} - y^2 - 8y \right) \Big|_1^4 \right] \\
&= \frac{8}{3} - \frac{8}{9}\left(\frac{64}{3} - 16 - 32 - \frac{1}{3} + 1 + 8 \right) = \frac{8}{3} - \frac{8}{9}(-18) = \frac{8}{3} + 16 = \frac{56}{3}.
\end{aligned}$$

And lastly, more complicated regions (those that are not elementary), can usually be broken up into a set of elementary regions that meet along boundaries. Then the integrals over these adjacent regions can be added together, noting that the contributions along the boundaries will be zero.

Example 14.6. Consider the *annular* region $\mathcal{D}$ between the two planar equations $x^2+y^2=1$ and $x^2+y^2=4$. This region is not elementary! But in the plane, slice up $\mathcal{D}$ into four regions using the two vertical lines $x=\pm 1$, as in Figure **??**. Then we have

- $\mathcal{D}_1 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; -4 \le x \le -1,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \right\}$,
- $\mathcal{D}_2 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; -1 \le x \le -1,\ -\sqrt{4-x^2} \le y \le -\sqrt{1-x^2} \right\}$,
- $\mathcal{D}_3 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; -1 \le x \le -1,\ \sqrt{1-x^2} \le y \le \sqrt{4-x^2} \right\}$,
- $\mathcal{D}_4 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; 1 \le x \le 4,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \right\}$. As written, all of these are Type I.

Example 14.7. ***Find the area of a circle of radius*** $r \ge 0$. Here, the circle of radius r centered at the origin is the set of points that satisfy the equation $x^2+y^2=r^2$. The region is then the *closed disk* $\mathcal{D}_r$ consisting of the interior of this circle and the circle itself. It is elementary of Type III, and can be written as elementary of Type I as

$$\mathcal{D}_r = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; -r \le x \le r,\ -\sqrt{r^2-x^2} \le y \le \sqrt{r^2-x^2} \right\}.$$

Using this, then, we have

$$\text{Area}(\mathcal{D}_r) = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \, dy \, dx.$$

Two things here: First, what is the integrand here? And why does this work? And secondly, finish this calculation. Note that you will have to use a inverse trig substitution to solve this. Perhaps THAT is why you will wind up with the answe: $\text{Area}(\mathcal{D}_r) = \pi r^2$.