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LECTURE 10: VECTOR FIELDS.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. Vector fields, as geometric objects and/or functions, provide a backbone in which all of physics and engineering, really mathematical modeling is structured on. From force fields in physics to slope fields in differential equations and modeling, the notion of a vector field allows us to recover measureable quantities from models defined only by equations of motion. Here, we begin the study of their basic structure and properties.

Vector Fields. We start with a definition:

Definition 10.1. A *vector field* on $\mathbb{R}^n$ is a map $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$, as assignment of a vector $\mathbf{F}(\mathbf{x})$ to every point $\mathbf{x} \in X$.

Examples of vector fields include

- Force fields in physics,
- slope fields in differential equations, and
- fluid (air) flow in climate models.

A vector field is of class C^n precisely when $\mathbf{F}$ is C^n. This means that vectors vary in both size and direction in a continuous (C^0), or differentiable (C^n, $n \geq 1$), etc.

Definition 10.2. A vector field is called a *gradient field* on $\mathbb{R}^n$ if $\mathbf{F}$ is the gradient of a real-valued function $f : X \subset \mathbb{R}^n \to \mathbb{R}$.

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1. Then $\nabla f : X \subset \mathbb{R}^n \to \mathbb{R}^n$. Here, we interpret this as a vector field on X, a gradient field on X.

- Here, f is called a *potential function* for the gradient field $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$.
- Recall for $f : \mathbb{R}^2 \to \mathbb{R}$, the level sets of f are generically curves in the domain of f, which is the plane. For f a potential function of a gradient field,
  1. the level curves are *equipotential sets*, sets of equal potential, and
  2. the gradient field along these sets always is orthogonal to (the tangent lines of) these sets.
  3. The gradient field always points in the direction of the most rapid increase of f at each point.
  4. In contrast to a vector field, a real-valued function is sometimes called a *scalar field*. The gradient takes a potential (scalar) field to a vector field.

Example 10.3. Given a scalar field $f : X \subset \mathbb{R}^n \to \mathbb{R}$, finding its gradient field is straightforward: Take derivatives adn form the vector. But, given a gradient field, can one find a potential for it? In Example 5 on page, 231, the author tells you that the gradient field $\mathbf{F}(x, y, z) = \begin{bmatrix} 3x^2 + y^2 \\ 2xy \\ x^3 - 2z \end{bmatrix}$ has potential $f(x, y, z) = x^3 z + xy^2 - z^2$, and "leave[s] it to you to verify...". But without a candidate for a potential, how does one calculate one? The idea is to un-differentiate!

Pretend that $f:\mathbb{R}^n \to \mathbb{R}$ is unspecified. But since we know $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$, then we know the following:

(1) $\dfrac{\partial f}{\partial y} = 2xy$. Hence $f(x,y,z) = xy^2 + h(x,z)$. (Why is this? Because, the partial derivative of f with respect to why would see every function of only x and z as a constant. Hence, when un-differentiating, one has to account for this fact by specifying the constant lost to differentiating (with respect to y) as something which is a function of possibly everything except for y. Got it? So we already know something about f. Namely, f is of the form $f(x,y,z) = xy^2 + h(x,z)$.

(2) Then $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left[xy^2 + h(x,z)\right] = y^2 + \dfrac{\partial h}{\partial x}(x,z) = 3x^2z + y^2$. This is bacause the last expression is the x-component of the gradient field $\mathbf{F}(x,y,z)$. Hence $h(x,z) = x^3z + g(z)$, where g is some unknown function of only z. Now we know even more about f. We know $f(x,y,z) = xy^2 + x^3z + g(z)$.

(3) And lastly, $\dfrac{\partial f}{\partial z} = \dfrac{\partial}{\partial z}\left[xy^2 + x^3z + g(z)\right] = x^3 + g'(z) = x^3 - 2z$. But this means that $g'(z) = -2z$, so that $g(z) = -z^2$.

Hence we have $f(x,y,z) = xy^2 + x^3z - z^2$.

Definition 10.4. A *flow line*, or a *trajectory* of a vector field $\mathbf{F}: X \subset \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable curve $\mathbf{x}: I \subset \mathbb{R} \to \mathbb{R}^n$ that satisfies

$$\mathbf{x}'(t) = \mathbf{F}\left(\mathbf{x}(t)\right), \quad \forall t \in I. \tag{10.1}$$

Here, the velocity vector of the curve at every point in the domain of the curve equals to vector field at that point. Finding such a path, given a vector field, is precisely the subject of a field of mathematics called differential equations! But simply verifying that a given path is a flow line of a vector field is a matter of just verifying Equation 10.1.

Example 10.5. Is the path $\mathbf{x}(t) = \begin{bmatrix} e^{-t} + 2e^{2t} \\ -e^{-t} + e^{2t} \end{bmatrix}$ in the plane a flow line for the vector field $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x + 2y \\ x \end{bmatrix}$ on $\mathbb{R}^2$?

The answer is yes, since

$$\mathbf{x}'(t) = \begin{bmatrix} -e^{-t} + 4e^{2t} \\ e^{-t} + 2e^{2t} \end{bmatrix} = \begin{bmatrix} (e^{-t} + 2e^{2t}) + 2\left(-e^{-t} + e^{2t}\right) \\ (e^{-t} + e^{2t}) \end{bmatrix} = \begin{bmatrix} x(t) + 2y(t) \\ x(t) \end{bmatrix} = \mathbf{F}\left(\mathbf{x}(t)\right).$$

Definition 10.6. An *linear operator* is a mapping from one linear (vector) space to another.

From linear algebra, this means that any matrix determines a linear operator from the domain to the codomain. This also means that any linear transformation is also called a linear operator. However, one can define linear spaces whose elements are functions, in the following way: Two real-valued functions, defined on the same space, can be added together to create another function from the same domain to the same codomain. Once can also multiple any function by a real number to create a new function. Hence any linear combination of real-valued functions from a domain to $\mathbb{R}$ is again a function from the domain to $\mathbb{R}$. And since there exists an additive identity function (the 0-function), and an additive inverse function for every function, the set of functions from a domain to $\mathbb{R}$ form a linear space (like a vector space). However, these "function" spaces are not finite dimensional, and hence there is not a finite basis, like for the standard vector spaces one sees in linear algebra. But one can define linear maps between these function spaces, and they behave much like the linear transformations you have seen in linear algebra. So think of linear operators as maps taking functions to functions. Note that the notion of an operator being *linear* is just the idea that the image of a linear combination of inputs is just a linear combination of the images of the inputs, or

$$f(c_1x + c_2y) = c_1f(x) + c_2f(y).$$

Keep this in mind.

Definition 10.7. The *del operator* ∇ is the linear operator that takes a real-valued C^1-function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ to its gradient vector field $\nabla f : X \to \mathbb{R}^n$.

Some Notation: $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ in $\mathbb{R}^3$, or

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \ldots + \mathbf{e}_n\frac{\partial}{\partial x_n}, \quad \text{in } \mathbb{R}^n.$$

This notation may seem a bit odd, but it is common, and implies

$$\nabla(\ ) = \sum_{i=1}^{n} \mathbf{e}_i\frac{\partial}{\partial x_i}(\ ) = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}(\ )$$

is to be interpreted as

$$\nabla f = \nabla(f) = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}(f) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}.$$

Definition 10.8. For a C^1-vector field $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$, the *divergence* of $\mathbf{F}$, denoted $\mathbf{div}\,\mathbf{F}$, or $\nabla \cdot \mathbf{F}$, is the scalar function

$$\mathbf{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \ldots + \frac{\partial F_n}{\partial x_n},$$

for $\mathbf{x} = (x_1, \ldots, x_n) \in X$, and $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} F_1(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{bmatrix}$.

Some notes:

- Here $\mathbf{div}\,\mathbf{F}(\mathbf{x}) = \nabla \cdot \mathbf{F}(\mathbf{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} \cdot \begin{bmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{bmatrix}$ uses the Dot Product, although the product on each component is meant to indicate "apply the partial operator to the component function".
- We will prove this later on in the course, but the divergence of a vector field measures the infinitesimal volume change caused by the vector field.
- A vector field $\mathbf{F}$, where $\nabla \cdot \mathbf{F} = 0$ is called *incompressible.*
- Viewed as an operator, ∇ can *operate* on functions in different ways:
  (1) As the gradient of a scalar field ∇f, for $f : X \subset \mathbb{R}^n \to \mathbb{R}$;
  (2) As the divergence of a vector field $\nabla \cdot \mathbf{F}$, for $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$; And
  (3) as the *curl* of a vector field $\nabla \times \mathbf{F}$, but only in $\mathbb{R}^3$.

Definition 10.9. For a C^1-vector field $\mathbf{F} : X \subset \mathbb{R}^3 \to \mathbb{R}^3$, the *curl* of $\mathbf{F}$, denoted $\mathbf{curl}\,\mathbf{F}$, or $\nabla \times \mathbf{F}$, is

$$\nabla \times \mathbf{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

More Notes:

- It is worth noticing that (1) the gradient of a scalar field is a vector field, (2) the divergence of a vector field is a scalar field, and (3) the curl of a vector field (in R^3) is a vector field.
- We will again prove this later, but the curl of a vector field measures the infinitesimal twist in the vector field along the vector field at each point.
- If, for $\mathbf{F} : X \subset \mathbb{R}^3 \to \mathbb{R}^3$, we have $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere, we say $\mathbf{F}$ is *irrotational.*

Example 10.10. The vector field $\mathbf{F}(x, y, z) = \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}$ rotates each xy-plane at $z = c$. Here

$$\nabla \times \mathbf{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$$
$$= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-x)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right)\mathbf{k} = -2\mathbf{k}.$$

Notice, by the definition and properties of the cross product, that, as a vector field $\nabla \times \mathbf{F}$ must be orthogonal to $\mathbf{F}$ at every point.

Example 10.11. Explosions and Implosions in $\mathbb{R}^3$: For $\mathbf{G}(x,y,z) = \pm c \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, for $c \in \mathbb{R}$, we have $\nabla \times \mathbf{G} = \mathbf{0}$. These vector fields are irrotational.

Exercise 1. Show all constant vector fields $\mathbf{H}(x,y,z) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ are irrotational.

And here are some beautiful facts, whose calculations provide excellent practice:

Exercise 2. Show that a gradient vector field in $\mathbb{R}^3$ is irrotational. That is, for $f : X \subset \mathbb{R}^3 \to \mathbb{R}$ a C^2-function, show that $\nabla \times (\nabla f) = \mathbf{0}$.

Exercise 3. Show that the curl of a vector field in $\mathbb{R}^3$ is incompressible. That is, for $F : X \subset \mathbb{R}^3 \to \mathbb{R}^3$ a C^2-vector field, show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.