January 28, 2019

LECTURE 1: PRELIMINARIES.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. This first lecture is just a bit of Linear Algebra backstory: As an introduction to the course, I thought to play with the structure of Euclidean space and linear algebra just to establish notation and begin the conversation. I also used a bit of Mathematica for visualization. It is listed on the course site.

Helpful Documents. Mathematica: `IntersectingPlanes`.

Real Euclidean Space $\mathbb{R}^n$. The real plane is often described as the set of all ordered pairs of real numbers. We can write this as

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \left\{ (x, y) \;\middle|\; x, y \in \mathbb{R} \right\}.$$

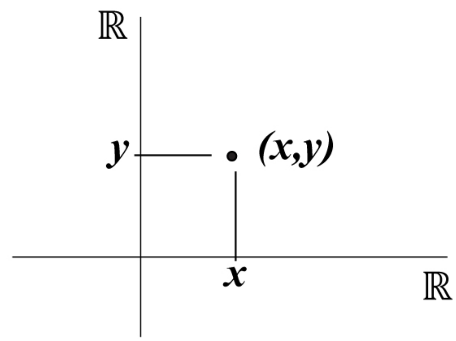

FIGURE 1.1. The plane $\mathbb{R}^2$.

The way the plane $\mathbb{R}^2$ is built out of two copies of the real line $\mathbb{R}$ is an example of a *Cartesian product*, a way of building a new set (called a product set) out of two sets, whose elements are pairs of elements of the two component sets, called factors, both $\mathbb{R}$ in this case. The set $\mathbb{R}^2$ is useful when studying functional relationships between sets because we can study the pairing given by the function as a subset living inside $\mathbb{R}^2$ by assigning the values of the input variable to the function x to one of the ordere3d pairs, and the output variable $y = f(x)$ to the other (See Figure 1.1). This gives us a visual depiction of the functional relationship which facilitates the study of its properties.

We can construct a form of addition in the set $\mathbb{R}^2$ by using the notion of addition in $\mathbb{R}$ and forming an addition in $\mathbb{R}^2$ component-wise:

$$(a, b) + (c, d) = (a + c, b + d).$$

With this addition (and the identity element $(0, 0)$ and an inverse $(-a, -b)$ for every set element (a, b)), we can turn $\mathbb{R}^2$ into a group. Here we would call $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ the *direct product* of the two groups $\mathbb{R}$. (A direct product is a Cartesian product on the underlying sets with whatever added structure the individual sets have and give to the product.) We can also multiply elements of $\mathbb{R}^2$ by real numbers (scalars multiplication), where

$$c \cdot (a, b) = (ca, cb)$$

and these two notions behave well together.

Now $\mathbb{R}$ is also a field, but $\mathbb{R}^2$ is not: One cannot construct a good notion of multiplication in $\mathbb{R}^2$ that satisfies all of the field axioms. However, with the notion of addition of ordered pairs, along with scalar multiplication, we can give $\mathbb{R}^2$ the structure of a *vector space* over $\mathbb{R}$.

Definition 1.1. A *linear* or *vector space* over a field is a set V of objects together with two operations which can be added together and multiplied by field elements in a "compatible" way.

It is common, in a linear space, to call the individual set elements "vectors". We also say that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$. But it will be a good idea to make a very important distinction:

Using Figure 1.2 as a guide, we will distinguish between points in $\mathbb{R}^2$, given by all 2-tuples of numbers written as

$$\mathbb{R}^2 = \left\{ p = (x,y) \;\middle|\; x,y \in \mathbb{R} \right\},$$

and vectors in $\mathbb{R}^2$, denoted as the set of all possible 2×1-matrices, or 2-vectors

$$\mathbb{R}^2 = \left\{ \mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; x,y \in \mathbb{R} \right\}.$$

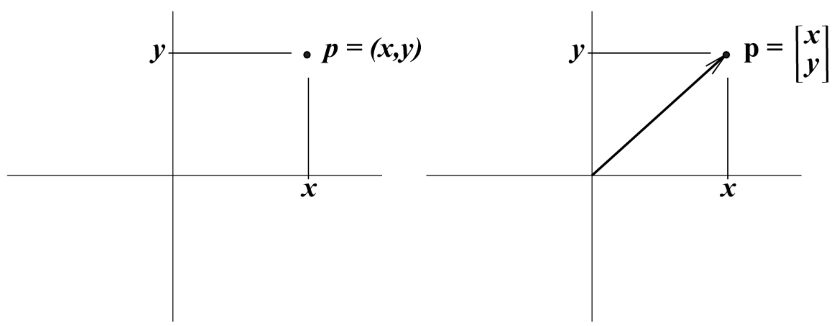

FIGURE 1.2. Points versus vectors, as elements of $\mathbb{R}^2$.

Some notes:

- Technically speaking, these two descriptions of the plane are quite different, even as there are equivalent. Note that this is a mathematical term that does need defining. For now we will leave it as is.
- In time, we will need to be able to define vectors based at arbitrary points in $\mathbb{R}^2$. Noticing a difference between points and vectors (with the same entries) as descriptions of the elements of the plane will help greatly later on when we define and understand vector fields.
- We can add still more structure to $\mathbb{R}^2$; a notion of a *scalar product*, sometimes called a *dot product* or an *inner product* on vectors (equivalently points):

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd \in \mathbb{R}.$$

  With this new structure, the plane becomes an example of an *inner product space*. This is very useful for vector spaces, since with this new structure, we can define notions of a distance between vectors, a vector's size, the angle between vectors, etc. And with these notions of measurement, the plane $\mathbb{R}^2$, as an inner product space, becomes a *Euclidean Space* (a space where one can do Euclidean geometry).

- Absolutely all of this still works with n-tuples of numbers: Define, for $n \in \mathbb{N}$,

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} = \left\{ x = (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, \text{ for } i = 0, 1, \ldots, n \right\}$$
$$= \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \; x_i \in \mathbb{R}, \text{ for } i = 0, 1, \ldots, n \right\}.$$

Now, a set of k n-vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are called *linearly independent* if for real scalars c_i, $i = 1, \ldots, k$,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \tag{1.1}$$

is only solved by $c_1 = c_2 = \ldots = c_k = 0$. If this is true, then none of the vectors can be written as a linear combination of the others.

Example 1.2. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ are *linearly dependent* since $3\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. Thus, for instance, one can write $\mathbf{v}_3$ as a linear combination of the others;

$$3\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_3.$$

If one can find n vectors that are linearly independent in $\mathbb{R}^n$, then this set of n vectors can act as a *basis*, in that any vector in $\mathbb{R}^n$ can then be written as a linear combination of these. So if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ are linearly independent (that is, if they form a basis), then

$$\mathbb{R}^n = \textbf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$
$$= \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \; \mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n, \; c_i \in \mathbb{R} \right\}.$$

Here, the term $\textbf{span}\{\cdot\}$ is just the set of all linear combinations of....

An interesting side note: Using $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ as a basis for $\mathbb{R}^n$, the lines through the origin formed by taking the set of all multiples of each vector $\mathbf{v}_i$ can serve as axes for a coordinate system on $\mathbb{R}^n$. Indeed, if each $\mathbf{v}_i$ serves as a unit of measurement (a measuring stick) on the line that it helps to create, then the c_is in any linear combination of basis vectors are the coordinates in that coordinate system, and different from what would be considered the standard one.

Example 1.3. Construct the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

These vectors form a basis of $\mathbb{R}^n$, since $\mathbb{R}^n = \textbf{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. This is called the *standard basis* for $\mathbb{R}^n$. See Figure 1.3 below.

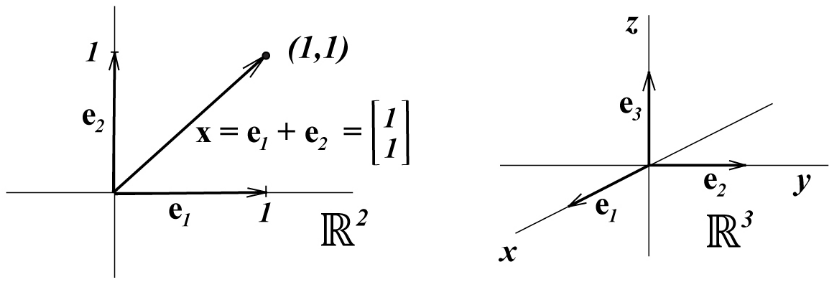

FIGURE 1.3. The standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Note that these standard bases are used to define the equivalence between the notion of $\mathbb{R}^n$ defined as points and the notion of $\mathbb{R}^n$ defined as n-vectors.

Definition 1.4. A *linear* or *vector subspace* W of a vector space V is a subset of the elements of V that satisfy

(1) $\mathbf{0} \in W \subset V$,
(2) If $\mathbf{w}_1, \mathbf{w}_2 \in W$, then $\mathbf{w}_1 + \mathbf{w}_2 \in W$, and
(3) if $\mathbf{w} \in W$, then for all $c \in \mathbb{R}$, $c\mathbf{w} \in W$.

It is good to note here that ALL vector subspaces pass through the origin (contain the zero-vector).

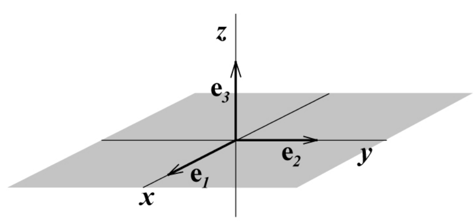

FIGURE 1.4. The xy-plane in R^3.

And going back to Equation 1.1, note that for any $k \in \mathbb{N}$, the set of n-vectors $\mathbf{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is ALWAYS a linear subspace of $\mathbb{R}^n$. How big it is as a subspace depends on the number of $\mathbf{v}_i$s are linearly independent.

Example 1.5. The set $\mathbf{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$ is commonly referred to as the xy-plane in $\mathbb{R}^3$, thinking of the standard coordinates in $\mathbb{R}^3$. The span of these three vectors only makes a plane in three space since the third vector is simply twice the first plus $3/2$ times the second. A basis for the span of these three 3-vectors can readily be the first two vectors in the standard basis of $\mathbb{R}^3$. Note that one can also call this linear subspace the $(z = 0)$-plane. In this way, the xy-plane is a version of $\mathbb{R}^2$ sitting inside $\mathbb{R}^3$ as a subspace of all vectors with 0 in the last component. See Figure 1.4.

Example 1.6. $\mathbf{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$ is a line passing through the origin in $\mathbb{R}^3$.

Example 1.7. Let $V = \mathbf{span}\left\{ \mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \right\}$. Then $V \subset \mathbb{R}^3$ is a 2-dimensional subspace, since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent (recall that the dimension of a (finite-dimensional) vector space is the number of elements in any basis), and $V \subset \mathbb{R}^3$ will look like a plane passing through the origin (See Figure 1.5, with **a** and **b** in red). The two 3-vectors

$$\mathbf{c} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

differ in that **c** (shown in blue in Figure 1.5 is IN the plane V, while **d** (shown in green in the figure) is not. Indeed, $\mathbf{c} = -\mathbf{a} + \mathbf{b}$, but there are not constants $c_a, c_b \in \mathbb{R}$, where $c_a\mathbf{a} + c_b\mathbf{b} = \mathbf{d}$. We would say that **d** is linearly independent from V.

Further, by Example 1.7, we can view the lines passing through **a** and **b** as coordinate axes for V, and on each axis, use the length of the contained vector as the unit length along that axis, marking, for example, 0 at the origin and 1 at the head of **a**. This provides a coordinate system directly on V, using the ordered pair (c_a, c_b) as the coordinates in V. Thus the vector $\mathbf{c} \in V \subset \mathbb{R}^3$ corresponds to the point $(3, 1, 2) \in \mathbb{R}^3$, but in the coordinates defined directly on V by the basis $\{\mathbf{a}, \mathbf{b}\}$, $\mathbf{c} \in V$ corresponds to the point $(-1, 1)$ in the *parameterization* of V given by the basis. The idea of placing coordinates directly on a subspace instead of using the ambient coordinates of the larger space is an important one. We will spend much time on this.

One way to describe a subspace like $V \in \mathbb{R}^3$ is through another form of multiplication of vectors, this one where the product of two 3-vectors is again a 3-vector. (Note that this is extremely rare and for now is limited to $\mathbb{R}^3$.) The cross product of two vectors $\mathbf{a} \times \mathbf{b} = \mathbf{n}$ is a vector normal (as in zero dot product) to both **a** and **b**. Hence, for any vector **n**, the set of all vectors normal to **n** is a two dimensional subspace $V \in \mathbb{R}^3$. And, if **n** is given as the cross product of two linearly independent vectors **a** and **b**, then **a** and **b** serve as a basis for V. Indeed, endow $\mathbb{R}^3$ with the coordinates x, y, and z. Then the equation

$$\mathbf{n} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 = n_x x + n_y y + n_z z,$$

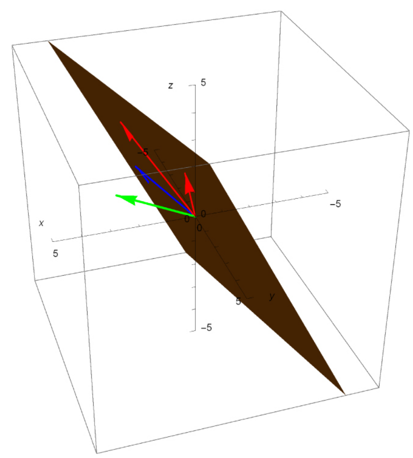

FIGURE 1.5. $V = \mathbf{span}\,\{\mathbf{a}, \mathbf{b}\}$.

defines a plane passing through the origin in $\mathbb{R}^3$. In Example 1.7, we have

$$\mathbf{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2(3) - 3(2) \\ -(1)3 + 3(2) \\ 1(-2) - 2(2) \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ -6 \end{bmatrix}.$$

Thus the vector (sub)space V is defined

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 12x + 3y - 6z = 0 \right\}.$$

Check for yourself that, for the vectors Example 1.7, $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$, but $\mathbf{d} \notin V$.

One conclusion that can be drawn from this is that one can define a plane in $\mathbb{R}^3$ via a single equation. But then, what is the equation of a line in $\mathbb{R}^3$?. Here is an example:

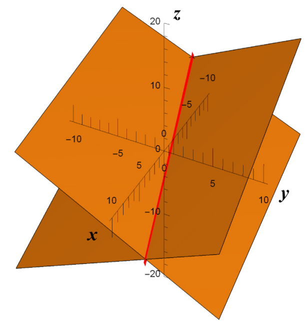

FIGURE 1.6. A line in $\mathbb{R}^3$ as the intersection of 2 planes.

Example 1.8. Consider the solution set for the set of equations:

$$\left.\begin{array}{rcll} x+2y+3z & = & 4 & \text{(eq1)} \\ 2x-2y+3z & = & 1 & \text{(eq2)} \end{array}\right\} \quad \text{2 equations in 3 unknowns.}$$

So what does this solution set in $\mathbb{R}^3$ look like? To see, solve as best as one can:

$$\begin{array}{rrcl} \text{(eq1)}+\text{(eq2)}: & 3x+6z & = & 5 \\ 2\text{(eq1)}-\text{(eq2)}: & 6y+3z & = & 7 \end{array}.$$

Then

$$x=\frac{5-6z}{3}, \quad y=\frac{7-3z}{6}, \quad z \text{ is free.}$$

Better yet, we can place a single parameter t directly on this set by setting $z=t$, so that $x=\frac{5-6t}{3}$ and $y=\frac{7-3t}{6}$, along with $z=t$ makes a parameterized curve (a line) in $\mathbb{R}^3$. One could also write this as a function (using vector notation):

$$\mathbf{c}:\mathbb{R}\to\mathbb{R}^3, \quad \mathbf{c}(t)=\begin{bmatrix} \frac{5-6t}{3} \\ \frac{7-3t}{6} \\ t \end{bmatrix}.$$

Note that, in this parameterization, we still have 3 equations in 4 unknowns. Do you notice a pattern between the number of equations, the number of unknowns and the "size" of the space of solutions?

So, roughly speaking, a space V is called linear if any linear combination of two elements in V is still in V. So what, then, is a *linear function*?

Definition 1.9. A function $f:\mathbb{R}\to\mathbb{R}$ is called *linear* if

$$f(c_1x_1+c_2x_2)=c_1f(x_1)+c_2f(x_2), \quad \forall x_1,x_2\in\mathbb{R}, \quad c_1,c_2\in\mathbb{R}/$$

Notes:

(1) With appropriate changes, this works equally well for $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^m$.
(2) Using this definition, then, the function $f(x)=3x$ is linear, but the function $g(x)=3x+1$ is NOT! To see this,

$$\begin{aligned} g(2+3) &= g(5)=3(5)+1=16 \\ &\neq g(2)+g(3)=(3(2)+1)+(3(3)+1)=17. \end{aligned}$$

The issue here is that for a function to be linear, the origin of the domain (the input space) must be mapped to the origin of the output space, so that $f(0)=0$. But here $g(0)=1$. And thus, $g(x)$ is not linear. It is an example of an *affine* function, one that can be seen as a composition of a linear function and a translation.
(3) Let $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^m$ be linear. Then, given a basis $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ for the domain $\mathbb{R}^n$, we can write any $\mathbf{x}\in\mathbb{R}^n$ as

$$\mathbf{x}=c_1\mathbf{v}_1+\ldots+c_n\mathbf{v}_n.$$

Then, since $\mathbf{f}$ is linear, we have

$$\begin{aligned}\mathbf{f}(\mathbf{x}) &= \mathbf{f}(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) = c_1\mathbf{f}(\mathbf{v}_1) + \ldots + c_n\mathbf{f}(\mathbf{v}_n) \\ &= {}_m\left\{ \underbrace{\begin{bmatrix} | & & | \\ \mathbf{f}(\mathbf{v}_1) & \ldots & \mathbf{f}(\mathbf{v}_n) \\ | & & | \end{bmatrix}}_{n} \right. \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A_{m\times n}\mathbf{x}.\end{aligned}$$

Hence, any linear map between vector spaces can always be represented by a matrix.