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LECTURES: DIFFERENTIAL FORM AND GENERALIZED STOKES.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. During the last lectures of this course, I have decided to lecture on the structure of differential forms from the perspective of multi-linear algebra and n-forms on vector spaces. This is basically not done in the book. This allows me to give a much more foundational treatment of just what forms are and not just how they work. We learn their structure, how to integrate them and how to differentiate them, all with an eye toward what works regardless of the dimension. We show how many of the things we learned in the past, from the Product Rule and the Substitution Method in Calculus I to the Change of Variables Theorem and Fubini's Theorem in Calculus III, are all just examples of a much more general structure. We then finish with the Generalized Stokes' Theorem, and show how the various big theorems of Gauss, Stokes and Green are also simply particular examples. We end with the same result of the Fundamental Theorem of Calculus. In fact, one can easily say that the Generalized Stokes Theorem is just the Fundamental Theorem of Multivariable Calculus.

Some (multi-)linear algebra. To start, let V be a n-dimensional vector space on $\mathbb{R}$. Then the points $\mathbf{v} \in V$ are called vectors, where

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad v_i \in \mathbb{R}, \quad \forall i = 1, 2, \dots, n.$$

A *linear functional*, or a *linear 1-form*, or a *covector* is a linear map $f : V \to \mathbb{R}$, where

$$f(a\mathbf{v} + b\mathbf{w}) = af(\mathbf{v}) + bf(\mathbf{w}), \quad \forall \mathbf{v}, \mathbf{w} \in V, \quad \forall a, b \in \mathbb{R}.$$

The set of all *covectors* of V is again an n-dimensional vector spacecalled the *dual space* to V, and denoted V^*. Note that, for the most part, we will stick with finte dimensional vector spaces.

What is a basis for V^*? Recall the standard basis for V, $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, where $\mathbf{e}_i$ is the n-vector all of whose entries is 0, except for the i entry, which is 1. Then, for each i, denote $\mathbf{e}_i^* : V \to \mathbb{R}$, the map $\mathbf{e}_i^*(\mathbf{v}) = v_i$ that strips off the i entry of $\mathbf{v}$. Note that this is a linear map on V. In this way, the set of linear maps $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \dots, \mathbf{e}_n^*\}$ form a basis for V^*, so that any covector (element of V^*) can be written as a linear combination of these:

$$\mathbf{v}^* = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \left(\sum_{i=1}^{n} \mathbf{e}_i^* \right) = v_1\mathbf{e}_1^* + v_2\mathbf{e}_2^* + \dots + v_n\mathbf{e}_n^*,$$

for $v_1, \ldots, v_n \in \mathbb{R}$. Then $\mathbf{v}^* : \mathbb{R}^n \to \mathbb{R}$ is a (finite-dimensional) operator, defined by

$$\begin{aligned}
\mathbf{v}^*(\mathbf{w}) &= v_1\mathbf{e}_1^*(\mathbf{w}) + v_2\mathbf{e}_2^*(\mathbf{w}) + \ldots + v_n\mathbf{e}_n^*(\mathbf{w}) \\
&= v_1 w_1 + v_2 w_2 + \ldots + v_n w_n \\
&= \mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w} \\
&= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.
\end{aligned}$$

Some notes:

- In this way, we often write linear functionals (covectors) as row vectors.
- The Dot Product

$$\mathbf{dot} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{dot}(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$$

  is not a linear function. It is on each factor, though, and is sometimes called a 2-linear function, and is an example of a *multilinear function*. One can create a version of the Dot Product function with one slot already filled. That is an example of a linear functional. Indeed, $\mathbf{v}^*(\ ) = \mathbf{dot}(\mathbf{v},\ )$.
- In $\mathbb{R}^3$, each $\mathbf{p} \in \mathbb{R}^3$ has a tangent space $\mathbb{T}_\mathbf{p}\mathbb{R}^3$, which is another copy of $\mathbb{R}^3$, but with its origin at $\mathbf{p}$. It is a *different* space!!

For coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}^n$, define a coordinate system on $T_\mathbf{p}\mathbb{R}^n$ as $(dx_1, \ldots, dx_n)$, where dx_i is the infinitesimal change in the x_i-direction at $\mathbf{p}$. Here, each dx_i is a linear functional on $T_\mathbf{p}R^n$ since, for $\mathbf{v} \in T_\mathbf{p}\mathbb{R}^n$, $dx_i(\mathbf{v}) = v_i$.

Notes:

- Think of a parameterized hypersurface $S \subset \mathbb{R}^n$, with $\mathbf{p} \in S$, and it is easier to see how a vector $\mathbf{v}$ tangent to S at $\mathbf{p}$ is actually in $T_\mathbf{p}S$ and not actually in S.
- This definition of dx_i, as a coordinate of the tangent space to R^n, works because coordinates themselves are actually linear functionals on a space (at least the Cartesian ones). They are projections onto the factors of $\mathbb{R}^n$ in a sense, which are linear functions.

To see this in more detail, let $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$. Then, endowing $\mathbb{R}^2$ with the coordinates x and y, we can write $x : \mathbb{R}^2 \to \mathbb{R}$, and $y : \mathbb{R}^2 \to \mathbb{R}$, defining them as the functions $x(\mathbf{p}) = p_1$, and $y(\mathbf{p}) = p_2$. These coordinate functions are linear functions and hence differentiable, with

$$Dx_\mathbf{p} : T_\mathbf{p}\mathbb{R}^2 \to \mathbb{R}, \quad Dx_\mathbf{p}(\mathbf{v}) = \mathbf{dot}(\mathbf{e}_1, \mathbf{v}) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1, \text{ and}$$

$$Dy_\mathbf{p} : T_\mathbf{p}\mathbb{R}^2 \to \mathbb{R}, \quad Dy_\mathbf{p}(\mathbf{v}) = \mathbf{dot}(\mathbf{e}_2, \mathbf{v}) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_2.$$

Now use this to define coordinates directly on $T_{\mathbf{p}}\mathbb{R}^2$, by (dx, dy), where

$$dx = Dx_{\mathbf{p}} = \mathbf{dot}(\mathbf{e}_1, \ \ ), \text{ and}$$
$$dy = Dy_{\mathbf{p}} = \mathbf{dot}(\mathbf{e}_2, \ \ ).$$

Example 20.1. Let $\mathbf{v} \in \mathbb{R}^3$ so that $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then the functions $x, y, z : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$x(\mathbf{v}) = \mathbf{dot}(\mathbf{i}, \mathbf{v}) = v_1,$$
$$y(\mathbf{v}) = \mathbf{dot}(\mathbf{j}, \mathbf{v}) = v_2, \text{ and}$$
$$z(\mathbf{v}) = \mathbf{dot}(\mathbf{k}, \mathbf{v}) = v_3$$

comprise the coordinates of $\mathbb{R}^3$. However, we often “abuse notation” for convenience and understandability and simply write

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

Using the above, we can write a linear functional on $\mathbb{R}^n$ as

$$\omega = a_1dx_1 + a_2dx_2 + \ldots + a_ndx_n = \mathbf{a} \cdot d\mathbf{x} = \mathbf{a}d\mathbf{x},$$

where $\mathbf{a}$ is the coefficient vector, and $d\mathbf{x} = \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}$ is the basis of coefficient covectors (corresponding to the coordinates) in $\mathbb{R}^n$.

Example 20.2. Let $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix}$. Then the linear functional on $\mathbb{R}^3$ corresponding to $\mathbf{a}$ acts on $\mathbb{R}^3$, and takes the vector $\mathbf{v} \in \mathbb{R}^3$ to

$$\begin{aligned} \omega(\mathbf{v}) &= a_1dx_1(\mathbf{v}) + a_2dx_2(\mathbf{v}) + a_3dx_3(\mathbf{v}) \\ &= a_1v_1 + a + 2v_2 + a_3v_3 = \mathbf{a} \cdot \mathbf{v} \\ &= 1(-4) + 2(-5) + 3(-6) = -32. \end{aligned}$$

Example 20.3. Also, keep in mind where each of these objects “live”: Let $\mathbf{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be a point in the plane and $\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \in T_{\mathbf{p}}\mathbb{R}^2$. Then while we envision $\mathbf{v}$ as a vector “in $\mathbb{R}^2$” based at $\mathbf{p}$, it is really a vector based at the origin of the tangent space $T_{\mathbf{p}}\mathbb{R}^2$ to the plane at $\mathbf{p}$.

Example 20.4. Let $I = [a, b] \subset \mathbb{R}$, and $\mathbf{c} : I \to \mathbb{R}^2$ be a C^1-curve. Since, for $\mathbf{p} \in \mathbf{c}(I) \subset \mathbb{R}^2$, $T_{\mathbf{p}}\mathbb{R}^2$ is not the same plane as $\mathbb{R}^2$ (it has different coordinates, with a different origin), we can write the tangent line $\ell_{\mathbf{p}}$ via the coordinates of $T_{\mathbf{p}}\mathbb{R}^2$, since $\ell_{\mathbf{p}}$ is the set of all tangent vectors to $\mathbf{c}$ at $\mathbf{p}$, so $\ell_{\mathbf{p}} \subset T_{\mathbf{p}}\mathbb{R}^2$. Hence the equation for $\ell_{\mathbf{p}}$ in $T_{\mathbf{p}}\mathbb{R}^2$ is:

$$dy = (\text{constant})dx.$$

So what is this constant?

Let $\mathbf{c} : [0,2] \to \mathbb{R}^2$ be defined by $\mathbf{c}(t) = (t, t^2)$. Then, in the xy-plane, the equation for the line $\ell_{\mathbf{p}}$ at $\mathbf{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is

$$(y-1) = 2(x-1), \quad \text{or} \quad y = 2x - 1.$$

However, in $T_{\mathbf{p}}\mathbb{R}^2$, the equation for the line $\ell_{\mathbf{p}}$ is $dy = 2dx$, or $\frac{dy}{dx} = 2$. Do you see where the notation for the derivative comes from now??