Section 7.2

In the same way that we integrated functions (real-valued) and vector fields over curves, we can do so over surfaces.

(I) Real-valued (scalar) functions

- Like for scalar line integrals, if the surface in $\mathbb{R}^3$ is inside the domain of a real-valued function in $\mathbb{R}^3$, we can restrict the domain to the surface and integrate.
- If we parameterize the surface with coordinates <u>on</u> the surface, this is then a double integral.
- However, the resulting ~~integral~~ value should be parameter independent.
- Basically, we look to define

$$\iint_{\text{surface}} f \, dS, \text{ where } dS = \|N(s,t)\| \, ds \, dt$$

is a surface differential.

<u>Def</u> Let $\mathbf{X}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth param. surface, where D is bounded. Let f be a C^0 function on a domain that includes $\mathbf{X}(D)$. Then the <u>scalar surface integral</u> of f along $\mathbf{X}$ is

$$\iint_{\mathbf{X}} f \, dS = \iint_D f(\mathbf{X}(s,t)) \, \| \mathbf{X}_s \times \mathbf{X}_t \| \, ds \, dt$$

$$= \iint_D f(x(s,t), y(s,t), z(s,t)) \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2} \, ds \, dt$$

<u>Notes</u>: ① Like for line integrals, dS is a scalar 2-form (ds is a scalar 1-form) and represents an infinitesimal element in surface area along the surface.

② For $f(x,y,z) = 1$, this integral gives the surface area of $\mathbf{X}(D)$.

③ In coordinates (s,t), this looks like a standard double integral.

④ If $\mathbf{X}$ is not smooth but has edges (piecewise smooth) then each smooth piece must be integrated separately and the results added together.

<u>Def.</u> Let $\vec{X}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth param. surface, where D is bounded. Let $\vec{F}$ be a C^0-vector field on a domain in $\mathbb{R}^3$ that includes $\vec{X}(D)$. Then the <u>vector surface integral</u> of $\vec{F}$ along $\vec{X}$ is

$$\iint_{\vec{X}} \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{X}(s,t)) \cdot \vec{N}(s,t)\, ds\, dt$$

$$= \iint_D \vec{F}(x(s,t), y(s,t), z(s,t)) \cdot \begin{bmatrix} \frac{\partial(y,z)}{\partial(s,t)} \\ -\frac{\partial(x,z)}{\partial(s,t)} \\ \frac{\partial(x,y)}{\partial(s,t)} \end{bmatrix} ds\, dt.$$

<u>Notes</u> ① Here $d\vec{S} = \vec{N}(s,t)\, ds\, dt$ is a vector 2-form. The differential of surface area written in terms of the normal to the surface at (s,t).

② If we normalize the normal vector

$$\vec{n}(s,t) = \frac{\vec{N}(s,t)}{\|\vec{N}(s,t)\|}, \text{ then}$$

$$\iint_{\vec{X}} \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{X}(s,t)) \cdot \vec{N}(s,t)\, ds\, dt = \iint_D \vec{F}(\vec{X}(s,t)) \cdot \vec{n}(s,t)\, \|\vec{N}(s,t)\|\, ds\, dt = \iint_{\vec{X}} (\vec{F} \cdot \vec{n})\, dS$$

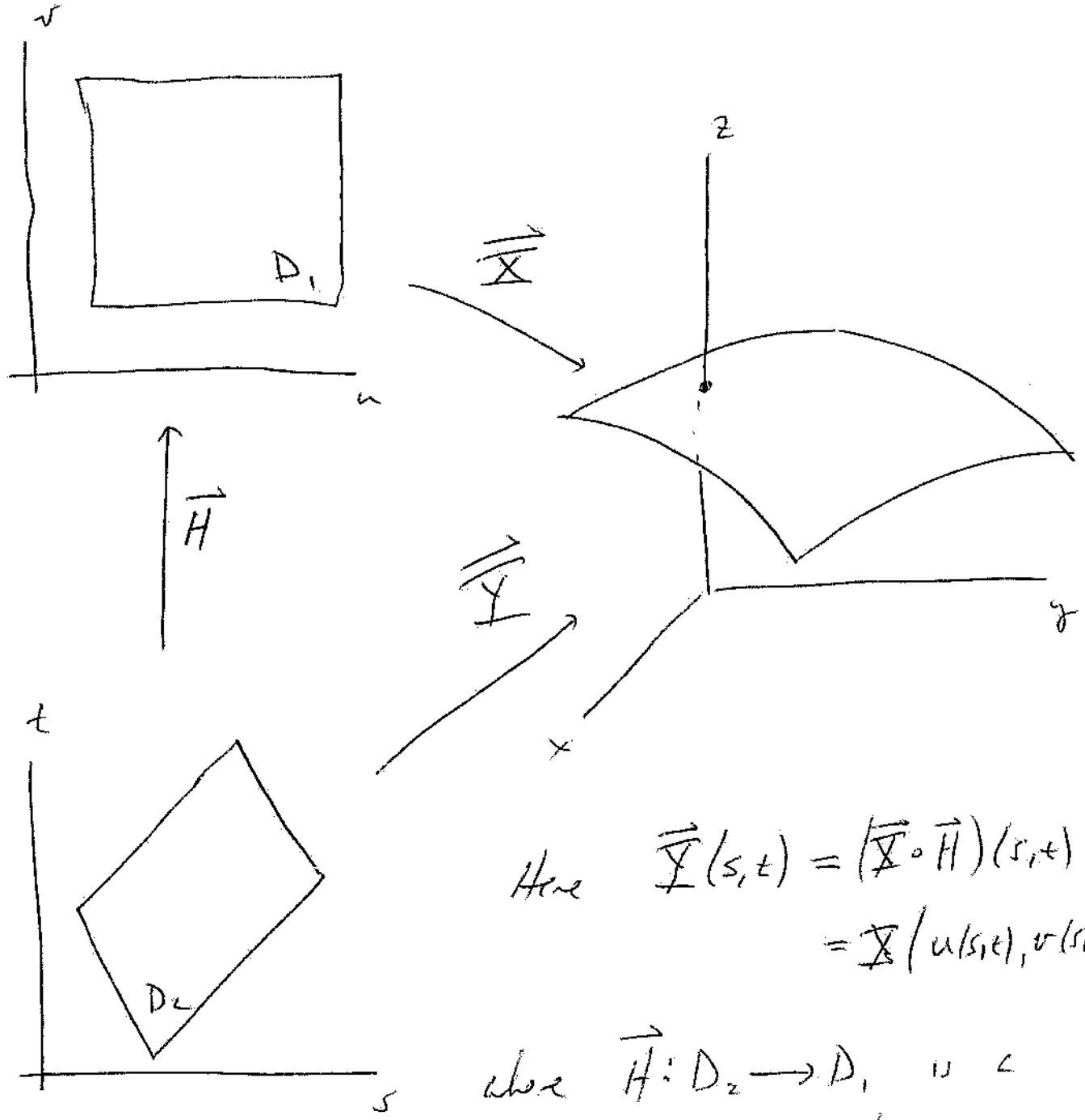
The vector surface integral of a vector field equals the scalar surface integral of the normal component of the vector field to the surface.

<u>Interpretation</u> – $\iint_{\vec{X}} \vec{F} \cdot d\vec{S}$ measures the vector field flow <u>through</u> the surface. This is called the <u>flux</u> of $\vec{F}$ through $\vec{X}(D)$.

Compare this to $\int_{\vec{X}} \vec{F} \cdot d\vec{s}$, the circulation, the vector field flow <u>along</u> $\vec{X}(D)$.

<u>Other facts</u>

(I) Given a parameterization $\vec{X}: D_1 \subset \mathbb{R}^2 \to \mathbb{R}^3$ and a C^0, 1-1, onto $\vec{H}: D_2 \to D_1$ with inverse $H^{-1}: D_1 \to D_2$, a <u>reparameterization</u> of $\vec{X}$ is $\vec{Y}: D_2 \to \mathbb{R}^3$, where $\vec{Y} = \vec{X} \circ H$.



Here $\vec{Y}(s,t) = (\vec{X} \circ \vec{H})(s,t)$

$= \vec{X}(u(s,t), v(s,t))$

where $\vec{H} : D_2 \longrightarrow D_1$ is a

bijective map w/ inverse, and

$H(s,t) = (u(s,t), v(s,t))$

A reparameterization is called smooth if both $\vec{X}$ and $\vec{Y}$ are and if H is C^1.

(II) <u>Thm</u> For f C^0 on a domain including a smooth $\vec{X}: D \to \mathbb{R}^3$, then for any smooth reparameterization $\vec{Y}$ of $\vec{X}$,

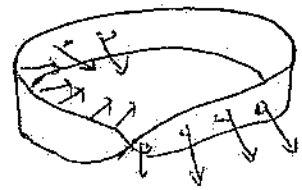
$$\iint_{\vec{Y}} f \, dS = \iint_{\vec{X}} f \, dS$$

(III) For a curve an <u>orientation</u> is a choice of continuously varying unit tangent vector along $\vec{c}$.

For a surface an <u>orientation</u> is a choice of continuously varying <u>unit normal vector</u> along $\vec{X}$ (above vs. below, inside vs outside).

<u>orientable</u>

vs



<u>non-orientable</u>

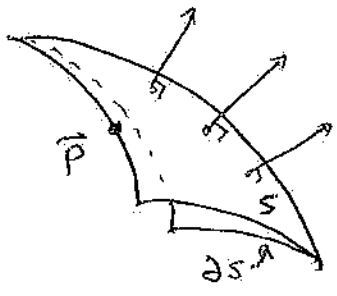
(IV) <u>Thm</u> If a reparameterization Ψ <u>preserves</u> <u>orientation</u> (i.e., Jacobian$(\vec{\Psi}) > 0$ everywhere),

$$\Longrightarrow \iint_{\vec{X}} \vec{F} \cdot d\vec{S} = \iint_{\vec{X}} \vec{F} \cdot d\vec{S}$$

otherwise introduce a minus sign to <u>RHS</u>.

Note: Recall $\vec{N}(s,t) = \vec{X}_s \times \vec{X}_t = -\vec{X}_t \times \vec{X}_s$.

(V) Orienting a surface automatically orients boundary curves on that surface.



Let S be an oriented surface with boundary in $\mathbb{R}^3$ & ∂S is a piecewise C^1 closed curve.

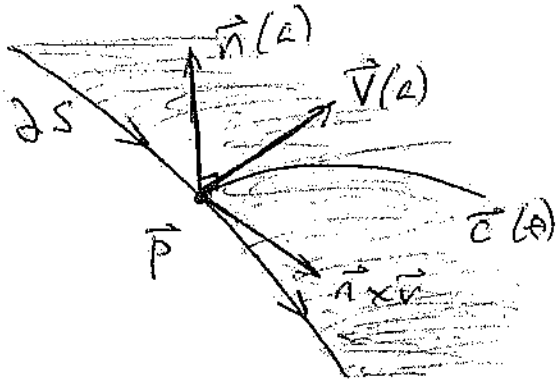
Let $\vec{p} \in \partial S$, where

$$\vec{p} = (s_0, t_0) = (x(s_0,t_0), y(s_0,t_0), z(s_0,t_0))$$

and choose $\vec{c} : [a,b] \longrightarrow S \subset \mathbb{R}^3$ a smooth curve such that $\vec{c}(a) = \vec{p}$, and $\vec{c} \cap \partial S = \{\vec{p}\}$.

Define $\vec{n}(\vec{p}) = \lim_{t \to a} \vec{n}(\vec{c}(t))$; and

$$\vec{v}(a) = \lim_{t \to a} \vec{c}\,'(t).$$



Here, $\vec{n}$ and $\vec{v}$ are based at $\vec{p}$ and are perpendicular. Hence they determine a 2-d subspace containing $\vec{v}$ and $\vec{n}$.

Then $\vec{n} \times \vec{v}$ is perpendicular to both and using the RHR determines a unique direction on ∂S.

This is the <u>direction</u> used in <u>Green's Thm</u>!