Section 2.5

The Chain Rule

Recall for $f, g : \mathbb{R} \to \mathbb{R}$, $f, g \in C^1$

$$\frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x)$$

In essence, the derivative of a composition is the product of the derivatives...,
with a twist - the derivative of the outside fnc is evaluated at the image of x under the inside fnc.

Let $f : J \subset \mathbb{R} \to \mathbb{R}$
$g : I \subset \mathbb{R} \to \mathbb{R}$.

Then the domain of $(f \circ g)$ is $\{x \in I \mid g(x) \in J\}$
$= g^{-1}(J) \cap I$.

ex. Let $f(x) = \sqrt{x}$, $g(x) = 2 - x^2$
Domain of f is $[0, \infty)$ and of g is $\mathbb{R}$.

Domain of $f \circ g$ is only where $2 - x^2 \geq 0$, or $[-\sqrt{2}, \sqrt{2}]$, as $(f \circ g)(x) = \sqrt{2 - x^2}$

Domain of $g \circ f$? $(g \circ f)(x) = 2 - x$
but domain is only $[0, \infty)$.

Note: In Leibniz notation, let $\underbrace{z=f(y),\ y=g(x)}_{z=f(g(x))}$.

Then $$\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}$$

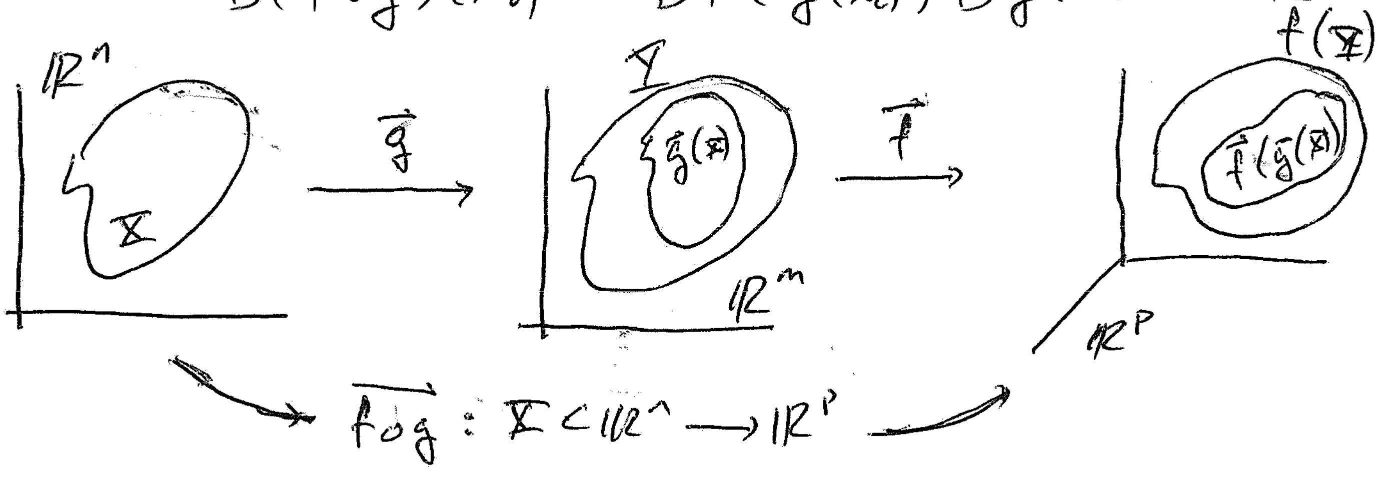
Careful though: At a point $x=a$.

$$\left.\frac{dz}{dx}\right|_{x=a}=\left.\frac{dz}{dy}\right|_{y=g(a)}\cdot\left.\frac{dy}{dx}\right|_{x=a}$$

In vector calculus, this still holds:

Thm 2.5.3 Suppose $X\subset\mathbb{R}^n$, $Y\subset\mathbb{R}^m$ are open, and $\vec{f}:Y\to\mathbb{R}^p$, $\vec{g}:X\to\mathbb{R}^m$ defined so that $\vec{g}(X)\subset Y$. If $\vec{g}$ is diff. at $\vec{x}_0\in X$, and $\vec{f}$ diff. at $\vec{y}_0=\vec{g}(\vec{x}_0)\in Y$, then $\vec{f}\circ\vec{g}$ is diff. at $\vec{x}_0$; with

$$D(\vec{f}\circ\vec{g})(\vec{x}_0)=Df(g(\vec{x}_0))\,D\vec{g}(\vec{x}_0)$$



ex. $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^3$, $\vec{f}(x,y) = (x^2y, 1, e^{xy})$.

$g: \mathbb{R}^3 \to \mathbb{R}$, $g(x,y,z) = xyz$.

Calculate $D(g\circ\vec{f})(x,y)$.

(I) Do composition first: $g\circ\vec{f}: \mathbb{R}^2 \to \mathbb{R}$.

$$g\circ\vec{f}(x,y) = g(\vec{f}(x,y)) = g(x^2y, 1, e^{xy}) = x^2ye^{xy}$$

$$D(g\circ\vec{f})(x,y) = \left[\frac{\partial(g\circ\vec{f})}{\partial x}(x,y) \quad \frac{\partial(g\circ\vec{f})}{\partial y}(x,y)\right].$$

$$= \left[2xye^{xy} + x^2y^2e^{xy} \quad x^2e^{xy} + x^3ye^{xy}\right].$$

(II) As a product of derivatives:

$$D\vec{f}(x,y) = \begin{bmatrix} 2xy & x^2 \\ 0 & 0 \\ ye^{xy} & xe^{xy} \end{bmatrix}.$$ and $Dg(x,y,z) = [yz \quad xz \quad xy]$.

so $Dg(f(x,y)) = Dg(x^2y, 1, e^{xy}) = [e^{xy} \quad x^2ye^{xy} \quad x^2y]$.

$\Rightarrow Dg(f(x,y)) \cdot D\vec{f}(x,y) =$

$$[e^{xy} \quad x^2ye^{xy} \quad x^2y]\begin{bmatrix} 2xy & x^2 \\ 0 & 0 \\ ye^{xy} & xe^{xy} \end{bmatrix} =$$

$$= \left[2xye^{xy} + x^2y^2e^{xy} \quad x^2e^{xy} + x^3ye^{xy}\right].$$

Now this all works, but be very careful V
with the variables!

Best to switch one set of variables.

$$\vec{f}(x,y) = (x^2y,\ 1,\ e^{xy})$$
$$g(u,v,w) = uvw$$

$$\Rightarrow\ Dg(u,v,w) = \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \end{bmatrix},\quad D\vec{f}(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix}$$

Now, in the composition, we know that

$$u = f_1(x,y) = x^2y$$
$$v = f_2(x,y) = 1$$
$$w = f_3(x,y) = e^{xy}$$

Here the derivative of the composition is

$$Dg(f(x,y)) \cdot D\vec{f}(x,y) = \begin{bmatrix} \frac{\partial(g\circ\vec{f})}{\partial x} & \frac{\partial(g\circ\vec{f})}{\partial y} \end{bmatrix}.$$

where

$$\frac{\partial(g\circ\vec{f})}{\partial x} = \frac{\partial g}{\partial u}\cdot\frac{\partial f_1}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial f_2}{\partial x} + \frac{\partial g}{\partial w}\cdot\frac{\partial f_3}{\partial x} \quad \Big\{ \text{Matrix multiplication}$$

$$= \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial g}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= vw\Big|_{\substack{v=1\\ w=e^{xy}}} \cdot (2xy) + uw\Big|_{\substack{u=x^2y\\ w=e^{xy}}} \cdot (0) + uv\Big|_{\substack{u=x^2y\\ v=1}} \cdot ye^{xy}$$

Here, the term $\frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial g}{\partial u}\Big|_{\vec{u}=f(\vec{x})} \cdot \frac{\partial u}{\partial x}\Big|_{\vec{x}}$

$$= \boxed{2xye^{xy} + x^2y^2e^{xy}} \checkmark$$

One more example

Let $\vec{c} : \mathbb{R} \to \mathbb{R}^3$ be a C^1-curve, and

$f : \mathbb{R}^3 \to \mathbb{R}$ a C^1-scalar valued fnc on $\mathbb{R}^3$.

Then $g = f \circ \vec{c} : \mathbb{R} \to \mathbb{R}$ can be viewed as

$f|_{\vec{c}}$. In this sense, ~~[illegible]~~

$g'(t) = \frac{df}{dt}(t)$ along $\vec{c}$. Calculate this.

Via the Chain Rule: $\vec{c}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ in C^1, and

$$\frac{d\vec{c}}{dt}(t) = \vec{c}\,'(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}, \text{ while}$$

$$Df(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}.$$

Hence $g'(t) = D(f \circ \vec{c})(t) = D\vec{f}\,(\vec{c}(t)) \cdot D\vec{c}(t)$

$$= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}$$

$$\underset{\text{along } \vec{c}}{\frac{df}{dt}(t)} = g'(t) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$\longrightarrow$ $\frac{d f|_{\vec{c}}}{dt}(t)$ or something like this.