EXTRA STUFF: A PRODUCT RULE FOR LIMITS WITH VECTOR-VALUED FUNCTIONS

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I mentioned in class that the notion of a limit of a function of more than one-independent variable can be tricky to study due to the problem of being able to "approach" a particular point $\mathbf{a} \in \mathbb{R}^n$ from so many ways when $n > 1$. However, I also talked about how limits, even when the domains and co-domains are bigger than $\mathbb{R}$, still behave much like their counter parts for scalar-valued functions of one independent variable (the Calculus I-II sort). For instance, the limit of a sum of functions at a point equals the sum of the limits of the summand functions at the point, *provided each of the summand limits exists*, that is. Limits of multiples of functions equals the multiple of the limit of a function also. And the limit of a product of functions is the product of the limits of the factor functions, *again provided that each factor limit exist.* The caveat in this last case, though, is that the product of two vector-valued functions must actually make sense for this statement to make sense at all. So, under the implicit idea that the product actually makes sense in this case, the Product Rule for vector-valued functions would in fact work. Let's look at some examples:

First, the book claims the scalar-valued function version of a product rule:

Theorem (Product Rule for Scalar-Valued Functions on $\mathbb{R}^n$)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, and suppose $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$ both exist. Then*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})g(\mathbf{x}) = \left(\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\right).$$

You can see how this may be a problem if the chosen functions above were vector-valued. Just how does one multiply together the output values of the two functions, given that they are vectors? The answer is that there are ways to multiply vectors together. Many, in fact. Does the Product Rule hold if we allow for such multiplications? In fact, it does:

Claim. *Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^p$, and suppose $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})$ both exist. Then*

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) \cdot \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})$$

when the products on each side are the same and when they make sense.

Proving this statement for ANY possible product is not necessary here. But we can show that it holds for the common products we use in this course. Here we show the rule for the Dot Product (defined when $m = p \in \mathbb{N}$). To start, recall Theorem 2.6 in the text, which states that for $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$, the limit $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \in \mathbb{R}^m$, where

$\mathbf{L} = \begin{bmatrix} L_1 \\ \vdots \\ L_m \end{bmatrix}$, if for $i = 1, \ldots, m$, we have $\lim\limits_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = L_i$. Here, the limit of a vector-valued function will exist iff each of its component scalar-valued functions has a limit, and that limit will be the vector of the component limits.

Proof when $m = p$, using the Dot Product. . Recall the Dot Product of two $\mathbb{R}^m$-vectors: For

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^m, \text{ then } \mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^m u_i v_i = u_1v_1 + u_2v_2 + \cdots + u_m v_m \in \mathbb{R}.$$

Now for $\mathbf{f}$ and $\mathbf{g}$ as above, we can use the dot product to write their product as a scalar-valued function

$$h : \mathbb{R}^n \to \mathbb{R}, \quad h(\mathbf{x}) = \mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x})g_i(\mathbf{x}).$$

Now assume that both $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$ and $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})$ exist, so that each of the limits of the component functions (of each factor) exist at $\mathbf{a}$. But this means

$$\begin{aligned}\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}) &= \lim_{\mathbf{x}\to\mathbf{a}} \sum_{i=1}^m f_i(\mathbf{x})g_i(\mathbf{x}) \\ &= \sum_{i=1}^m \lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x})g_i(\mathbf{x}) = \sum_{i=1}^m \left(\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{a}} g_i(\mathbf{x})\right)\end{aligned}$$

due to the product and sum rules for limits of scalar-valued functions. But notice that

$$\begin{aligned}\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})\cdot\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) &= \lim_{\mathbf{x}\to\mathbf{a}} \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix} \cdot \lim_{\mathbf{x}\to\mathbf{a}} \begin{bmatrix} g_1(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{x}) \end{bmatrix} \\ &= \begin{bmatrix} \lim\limits_{\mathbf{x}\to\mathbf{a}} f_1(\mathbf{x}) \\ \vdots \\ \lim\limits_{\mathbf{x}\to\mathbf{a}} f_m(\mathbf{x}) \end{bmatrix} \cdot \begin{bmatrix} \lim\limits_{\mathbf{x}\to\mathbf{a}} g_1(\mathbf{x}) \\ \vdots \\ \lim\limits_{\mathbf{x}\to\mathbf{a}} g_m(\mathbf{x}) \end{bmatrix} \\ &= \sum_{i=1}^m \left(\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{a}} g_i(\mathbf{x})\right).\end{aligned}$$

As they are equal, we are done. □

Exercise 1. *Show that this result also holds for $m = p = 3$ using the Cross Product.*

Question 1. *For a product of vectors to make sense, must it be the case that $m = p$?*

Exercise 2. *Show this result still holds for $m = 1$ and $p > 1$.*

Now create a matrix-valued function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ defined by

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f_{11}(\mathbf{x}) & F_{12}(\mathbf{x}) & \cdots & F_{1m}(\mathbf{x}) \\ F_{21}(\mathbf{x}) & F_{22}(\mathbf{x}) & \cdots & F_{2m}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ F_{p1}(\mathbf{x}) & F_{p2}(\mathbf{x}) & \cdots & F_{pm}(\mathbf{x}) \end{bmatrix}.$$

Really the co-domain here can be thought of as basically the real space of dimension pm (think of a 2×3-matrix as having 6 elements, just like a 6-vector). Assume that the limit of a function like $\mathbf{F}$ will exist at a point $\mathbf{a} \in \mathbb{R}^n$ iff ALL of the component functions have limits as scalar-valued functions, and the limit $\mathbf{L}$ will be an $p \times m$ matrix with elements L_{ij}.

Exercise 3. *Establish the Product Rule for this matrix-valued* $\mathbf{F}$ *and* $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ *where the product is standard matrix multiplication.*