EXAMPLE: LEVEL SETS OF A FUNCTION ON $\mathbb{R}^3$

110.211 HONORS MULTIVARIABLE CALCULUS
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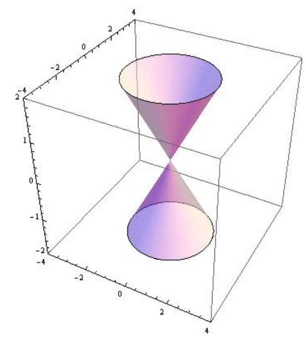
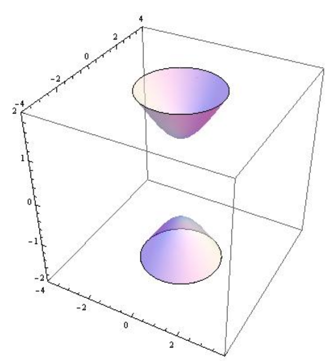
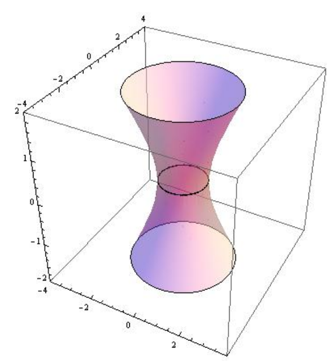


The set of solutions in $\mathbb{R}^3$ to the equation given by $z^2 = x^2 + y^2$ is called the *elliptic cone*, whose graph is given at right. Recognize that this is NOT the graph of a function $z = f(x, y)$, and even if we were to take ONLY the positive root when "solving" for z, the resulting graph would be different from that of the paraboloid, the function on two variables corresponding to the equation $z = x^2 + y^2$.

If we change the equation slightly by adding a constant to the right-hand side, the graph changes dramatically:

- The set $\left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, z^2 = x^2 + y^2 - 1\right\}$ is an example of a *one-sheeted hyperboloid* (are you also thinking of the cooling tower of a nuclear power plant?), while
- the set $\left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, z^2 = x^2 + y^2 + 1\right\}$ is called a *two-sheeted hyperboloid*.





Now consider the scalar-valued function

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z) = x^2 + y^2 - z^2.$$

The graph of f is best "seen" as a 3-dimensional hypersurface (space of size one-dimension less than the ambient space) of $\mathbb{R}^4$. Since we cannot, therefore, view it directly, we can study this function via its level sets. These turn out ro be surfaces living in the domain $\mathbb{R}^3$. What do they looks like?

For $c \in \mathbb{R}$, the set

$$f^{-1}(c) = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, c = x^2 + y^2 - z^2\right\},$$

or the solution set to the equation $z^2 = x^2 + y^2 - c$. Sets of this type are graphed above, no? We have

- $f^{-1}(0) =$ elliptic cone.

- $f^{-1}(0)\big|_{c<0}$ = 2-sheeted hyperboloid, and
- $f^{-1}(0)\big|_{c>0}$ = 1-sheeted hyperboloid.

Below are the graphs of a few of these level sets to give you a picture of how they look together.