HOMEWORK PRESENTATION MODEL

110.201 LINEAR ALGEBRA
PROFESSOR RICHARD BROWN

Welcome to the third week of class. I hope that the course and the material are as interesting and relevant to you and your interests as they are to me. Linear Algebra is a beautiful, yet fairly abstract high-level math course, and will be quite challenging. However, it will also be very rewarding both now and in the future as you take higher level courses in your chosen fields that require this way of thinking as prerequisite material. Stay active in this course, and it will go well.

The first Problem set has been submitted by you, graded by your TA, and returned to you. Please take time to do this following with this and future problem sets:

- Please look it over carefully and fully explore the nature of your strengths and any weaknesses in the set, both among the graded problems as well as those not graded. ALL of these are considered very relevant problems and will need to be well understood when it comes time for more consequential performance evaluation measures like exams. There is also, up on the course website, a list of what I call "Challenge Problems", really just another set of homework exercises that I find particularly interesting. There will not be any means to submit for evaluation these problems, and no solutions given. But they are good problems to further your studies and understanding of the subject. Please treat them seriously.
- Take the time to rewrite every graded solution on which you did not score full marks. Having a perfect working model for each of these problems is an excellent way to create a database of techniques and concepts for the course. And
- Compare your work with friends and collaborators in the class, both for the correctness of the other problems as well as to generate a better understanding of how others approach the same problems.

Remember that just because this problem set is complete, it is not the case that this material is done with. Math in general, and Linear Algebra in particular, is cumulative in the sense that a thorough understanding of future material will always depend critically on a firm grasp of past and present stuff. Every house can only stand on a sound foundation, eh?

Below are some examples of submitted solutions to one of the problems of the first problem set. The problem was Exercise 1.1.18, which is Chapter 1, Section 1, problem number 18. These are *actual* submissions, rewritten in my hand for the purpose of anonymity, and offered without edits as a complete solution.

Date: September 19, 2013.

Here is the first submission:

1.1 18) $\begin{bmatrix} x+2y+3z=a \\ x+3y+8z=b \\ x+2y+2z=c \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 1 & 3 & 8 & b \\ 1 & 2 & 2 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 5 & b-a \\ 0 & 0 & -1 & c-a \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & -7 & \begin{matrix} a-2b-2a \\ 3a-2b \end{matrix} \\ 0 & 1 & 5 & b-a \\ 0 & 0 & 1 & c-a \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \begin{matrix} 3a-2b+7(c-a) \\ 10a-2b-7c \end{matrix} \\ 0 & 1 & 0 & \begin{matrix} b-a-5a+5c \\ -6a+b+5c \end{matrix} \\ 0 & 0 & 1 & a-c \end{array}\right]$

$$\boxed{\begin{aligned} x &= 10a-2b-7c \\ y &= -6a+b+5c \\ z &= a-c \end{aligned}}$$

I would like you to critically think about this submitted solution to the problem by answering the question: ***What is the "value" of this solution?"***. Answer this question by considering the following criteria:

- How well does this solution answer, or even address, the actual question offered in the problem? Really, what is the question this solution is addressing?
- How can we, as a reader, "see" the thought process of the writer? The reader shouldn't have to read between the lines to establish the train of thought of the writer. It is the writer's duty to expose the writer's thought process to the reader.
- How correct is this solution? Are the methods of analysis utilized to understand the nature of the problem valid and relevant here? And, via a set of calculations, assumptions and deductions, is the logical progression of thoughts and calculations correct mathematical reasoning? It should always be understood that the actual final answer of a problem like this has almost no value whatsoever. The purpose of the problem is to expose the *process* with which you answer the question. The actual answer to the question doesn't really matter.
- Can this problem solution stand as a good working model for all problems of a similar type as this one, or that address a similar concept? Can you make a mental image of this solution and place it in your mind "next to" the question being asked? In this case, once the writer of this solution closed the book or moved onto another problem, the answer is obviously no. As a future model for review and further study, this solution says almost nothing about what the writer was thinking at the time the writer wrote the solution. The writer, when trying to again use this solution, would have to completely recreate the entire experience, which only lies between the lines here, and is lost since it was never documented.

Every problem offered as a study tool for the course is offered because it is exposing an aspect or concept or technique or detail of the theory relevant to the topic. Your solution is a means to identify what the point of this problem is in this context and use your answer to establish how this new view relates to the core of the course. Does this answer succeed in this respect in any way, shape or form?

Here is another solution:

1.1 18) Here we have 3 equations and 3 variables. Our goal, through multiplying, subtracting and adding the rows will be to isolate each variable.

$$
\begin{array}{l} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array}
\left|\begin{array}{l} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{array}\right|
\begin{array}{l} \\ (-\mathrm{I}+\mathrm{II}) \\ (-\mathrm{I}+\mathrm{III}) \end{array}
$$

$$
\Rightarrow \left|\begin{array}{r} x + 2y + 3z = a \\ y + 5z = -a+b \\ -z = -a+c \end{array}\right|
\Rightarrow \left|\begin{array}{r} x + 2y + 3z = a \\ y + 5z = b-a \\ z = a-c \end{array}\right|
\begin{array}{l} (-3\mathrm{III}+\mathrm{I}) \\ (-5\mathrm{III}+\mathrm{II}) \\ \end{array}
$$

$$
\Rightarrow \left|\begin{array}{r} x + 2y = 3c-2a \\ y = 5c+b-6a \\ z = a-c \end{array}\right|
\begin{array}{c} (\mathrm{I}-2\mathrm{II}) \\ \Rightarrow \\ \end{array}
\left|\begin{array}{r} x = 3c-2a-2(5c+b-6a) \\ y = 5c+b-6a \\ z = a-c \end{array}\right|
$$

$$
\Rightarrow \left|\begin{array}{r} x = 10a-2b-7c \\ y = -6a+b+5c \\ z = a-c \end{array}\right|
$$

- From this linear system, we see that the planes defined by the 3 equations intersect at the pt $(x,y,z) = (10a-2b-7c, -6a+b+5c, a-c)$
- These three planes intersect at this unique pt.

Go back a page and readdress ALL of the questions I posed on the first submitted solution, comparing both this one and the prior one to each other. Do you see that this solution has more merit than the other? It still does not fully address the question in a fundamental way, since we still do not really know what the question actually was. But there is a process here, a notion that the writer has a process and is telling you what is that process. And there is an attempt to actually answer a question. Carefully study the last two homework submissions.

And now a final, third submission:

1.1.18) Find all solutions to the linear system

$$\left|\begin{array}{l} x+2y+3z=a \\ x+3y+8z=b \\ x+2y+2z=c \end{array}\right|$$

where a, b, c are arbitrary constants.

Strategy: By creatively replacing eqns here with sums of multiples of other equations, we can create a system with the same solutions but with equations easier to solve. We seek to create a system where each equation only contains 1 variable of $\{x, y, z\}$ each.

Solution Label the equations

$$\begin{array}{ll} (\mathrm{I}) & x+2y+3z=a \\ (\mathrm{II}) & x+3y+8z=b \\ (\mathrm{III}) & x+2y+2z=c \end{array}$$

Using book notation, we have

$$\left|\begin{array}{l} x+2y+3z=a \\ x+3y+8z=b \\ x+2y+2z=c \end{array}\right| \begin{array}{l} \\ -(\mathrm{I})+(\mathrm{II}) \\ -(\mathrm{I})+(\mathrm{III}) \end{array} \longrightarrow \left|\begin{array}{r} x+2y+3z=a \\ y+5z=b-a \\ -z=c-a \end{array}\right|$$

And

$$\left|\begin{array}{r} x+2y+3z=a \\ y+5z=b-a \\ z=a-c \end{array}\right| \begin{array}{l} -3(\mathrm{III})+(\mathrm{I}) \\ -5(\mathrm{III})+(\mathrm{II}) \\ \longrightarrow \end{array} \left|\begin{array}{lr} x+2y & =3c-2a \\ y & =-6a+b+5c \\ z & =a-c \end{array}\right|$$

And

$$\left|\begin{array}{lr} x+2y & =3c-2a \\ y & =-6a+b+5c \\ z & =a-c \end{array}\right| \begin{array}{l} -2(\mathrm{II})+(\mathrm{I}) \\ \longrightarrow \end{array} \left|\begin{array}{lr} x & =10a-2b-7c \\ y & =-6a+b+5c \\ z & =a-c \end{array}\right|$$

For any choice of constants a, b, c, the set of all solutions to the system is given by

$$\left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x=10a-2b-7c \\ y=-6a+b+5c \\ z=a-c \end{array} \right\}$$

Again, compare these three solutions using ALL of the questions we addressed after the first solution. Think carefully about each question and study these three submissions. A proper model for *Homework Presentation* is much more important than calculating solutions. Think also about this:

- This last homework submission will take a LOT longer to prepare than the former two. But this form of homework preparation *IS STUDYING* in its most pure form. There will be little need for further preparation for exams and such if you adopt the final submission above as your model for homework preparation.
- This last homework submission will establish exactly the well-organized approach to mathematics that leads to a real understanding of the material, instead of merely an identification of topics, techniques and concepts. As a Scholar-in-Training, this is what your goal should be.

And one last point: This last submission was not from a student. It was from me. Sorry about that. I wanted to show you what I feel is a well-crafted response. Feel free to establish your own style. But take to heart this essay. See you in class.