110.106 CALCULUS I: BIOLOGICAL AND SOCIAL SCIENCES

Fall 2009
A Homework Design Example: One way

Suppose the current homework assignment (from the text *Calculus for Biology and Medicine*, by Claudia Neuhauser (Pearson Education, 2004) is as follows:

Section 3.2: Problems 6, 8, 16, 22, 23, 28, ... Here are the first two problems as stated:

6. Show that
$$f(x) = \begin{cases} \frac{2x^2+x-6}{x+2} & x \neq -2 \\ -7 & x = -2 \end{cases}$$
is continuous at $x = -2$.

8. Let
$$f(x) = \begin{cases} \frac{x^2+x-2}{x-1} & x \neq 1 \\ a & x = 1 \end{cases}$$
Which value must you assign to a so that $f(x)$ is continuous at $x = 1$.

On the next page are a representative sample of how I have often seen solutions written up for these two problems. Check them out.

Date: November 5, 2009.

Representative Homework Solution samples for this set: This is how I often see homework problem solutions written up for submission for a grade.

6) $f(x) = \begin{cases} \frac{2x^2+x-6}{x+2} & x \neq -2 \\ -7 & x = -2 \end{cases}$

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{2x^2+x-6}{x+2} = \lim_{x\to -2} \frac{(2x-3)\cancel{(x+2)}}{\cancel{x+2}} = \lim_{x\to -2} 2x-3 = -7 \quad \boxed{\text{continuous}}$$

8) $f(x) = \begin{cases} \frac{x^2+x-2}{x-1} & x \neq 1 \\ a & x = 1 \end{cases}$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2+x-2}{x-1} = \lim_{x\to 1} \frac{(x+2)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1} x+2 = 3 = a \quad \boxed{a=2}$$

Some comments here: First, if you were to use this problem set to study for an exam, you would need to understand the context of each problem in order to be able to use the problems as a guide to answering particular problem types. So ask yourself:

- **What is Problem 6 asking you to do, specifically?**
- **What was your strategy for solving the problem?**
- **How do you know when you have solved the problem and you can stop?**
- **Did you use any particular theorems or general theory to solve this problem?**

Weeks from now, when you are studying for the exam, and want to "go over" your homeworks, will you remember all of the answers to all of these questions? Or will you have to reconstruct the context for solving Problem 6? For example, you will need your book to remember what the question was asking or how it was stated. You will need to go back to the section to remember any theorems you used or theory you applied. You will need to remember the entire environment that you were in when you actually did this problem. Not only that, if this problem showed up on the exam, you would then have to put all of this stuff back together BEFORE you actually could use this problem as an example for the one on the exam. That takes up a LOT of time, no? Do you really want to do all of that during an timed exam when there are possibly 5 or 6 other problems still untried? Or would it be better to have a ready mental image of a well-constructed problem solution, with all of the above questions apparent in detail, ready to draw on during the test? An actual image of a detailed problem solution type, complete with context, strategy, calculations and conclusions in your head? Would be nice not to have to think too much on the exam, right? On the next page, I will show you what my solutions would look like....

Homework Solutions for this set: My take on what constitutes a well-constructed homework problem solution.

6) For $f(x) = \begin{cases} \frac{2x^2+x-6}{x+2} & x \neq -2 \\ -7 & x = -2 \end{cases}$, show $f(x)$ is continuous at $x = -2$.

Solution: By definition, A function $f(x)$ will be continuous at a point $x = a$ if $\lim_{x\to a} f(x) = f(a)$. So to solve this problem, we will need to show that $\lim_{x\to -2} f(x) = f(-2) = -7$. This would be like "plugging the hole" in the graph of $f(x)$. Here

$$\begin{aligned} \lim_{x\to -2} f(x) &= \lim_{x\to -2} \frac{2x^2+x-6}{x+2} \\ &= \lim_{x\to -2} \frac{(2x-3)\cancel{(x+2)}}{\cancel{x+2}} && \text{[you can cancel like factors in a limit]} \\ &= \lim_{x\to -2} 2x-3 && \text{[this is cont. at } x=-2\text{, so plug in directly]} \\ &= 2(-2)-3 = -7 = f(-2). \end{aligned}$$

Hence, $\lim_{x\to -2} f(x) = f(-2)$, and $f(x)$ is continuous at $x = -2$.

8) For $f(x) = \begin{cases} \frac{x^2+x-2}{x-1} & x \neq 1 \\ a & x = 1 \end{cases}$, choose a value for a so that $f(x)$ is continuous at $x = 1$.

Solution: Like Problem 6 above, $f(x)$ will be continuous at $x = 1$ if we choose the value of a so that $\lim_{x\to 1} f(x) = f(1)$. So here, we calculate $\lim_{x\to 1} f(x)$ and set $\lim_{x\to 1} f(x)$ equal to $a = f(1)$. Then $f(x)$ will be continuous at $x = 1$.

$$\begin{aligned} \lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{x^2+x-2}{x-1} \\ &= \lim_{x\to 1} \frac{(x+2)\cancel{(x-1)}}{\cancel{x-1}} && \text{[like Problem 6 above]} \\ &= \lim_{x\to 1} x+2 && \text{[again, this is cont. at } x=1\text{, so...]} \\ &= (1)+2 = 3. \end{aligned}$$

Hence, set $a = 3$. Then $\lim_{x\to 1} f(x) = f(1) = 3$ and $f(x)$ is continuous at $x = 1$.

Can you see the difference here? First, you will no longer need the text to see what the question actually was. Second, all of your thoughts, strategies, and concepts are listed IN the solution, so there is no need to reconstruct what you were doing when you actually solved the problem. Third, any mistakes you will make will be easy to see and easy to correct, since there is so much detail WITHIN your solution. later, when you come across a problem like this one, you will actually envision this solution, with all of its detail. There will be less need to actually "think" to solve the new one. In detail, here are the elements that make these solutions different from the ones on the previous page:

- **The problem has been explicitly stated IN the solution. You do not have to actually write out the problem word-for-word. But state enough detail that you know without a doubt what is being asked of you.**
- **The solution begins with a statement of background, strategy for actually solving the problem, and some of the theory you will use in solving the problem. This is called**

context, and greatly enhances comprehension, both now and in the future. It organizes your thoughts, and places the calculations within a course of action.

- The steps of any calculations are annotated with reasons for passing from one step to the other. Now you know what you were thinking as you were solving the problem. And any mistakes in reasoning will be easily seen and rectified.
- The conclusion re-addresses the statement of the problem. You know you have finished, since you answered the question.

Think of each solution as an essay in mathematical language. When you write an essay, you begin with a statement of your purpose for the essay. You introduce your position and state your strategy for arguing your point. This is the introduction. In the body, you elaborate on your points, and detail your arguments. In the conclusion, you restate your intentions, how your arguments reinforced your points, and what the repercussions of your statement may be. This solution then becomes a well-reasoned argument written in the language of mathematics. Really, this is all that it is.

One last point. This style of problem solving takes more time to do than the previous page. But the savings gained from not having to reconstruct the thinking behind half-baked solutions later on, combined with the savings in study time near the exam (since you ARE studying while you are doing homework!), combined with the mental images you will actually form in your head of the solutions of particular problem types, will result in a huge time savings in the long run. A common critique offered to students when learning music is (This is a quote from my son's piano teacher):

"Practice slowly to learn quickly"