HOMEWORK PROBLEM SET 9: DUE APRIL 14, 2017

110.302 DIFFERENTIAL EQUATIONS
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Question 1. For the following systems, find a general (real) solution, draw a direction field and plot enough trajectories to fully characterize the nature of the solutions to the system.

(a) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \mathbf{x}$.

(b) $\mathbf{x}' = \begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix} \mathbf{x}$.

(c) $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x}$.

Question 2. Solve the IVP

$$\mathbf{x}' = \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

and, in detail, describe the behavior of the solution, both near $t = 0$ as well as when t goes to infinity and minus infinity. For instance, describe how close it gets to the origin and any possible asymptotes that it may have. If it has asymptotes, describe the nature of these asymptotes.

Question 3. Find the general (real) solution to the ODE system

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \mathbf{x}.$$

Question 4. For the ODE system

$$\mathbf{x}' = \begin{bmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{bmatrix} \mathbf{x},$$

do the following:

(a) Solve the system.

(b) Choose a non-zero initial vector and sketch the trajectory in the x_1x_2-plane.

(c) For this trajectory, also sketch $x_1(t)$ and $x_2(t)$ as functions of t.

(d) For this trajectory, also sketch it in tx_1x_2-space.

Question 5. For the ODE system
$$\mathbf{x}' = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \mathbf{x},$$
do the following:

(a) Determine the eigenvalues as functions of α.

(b) Find the critical values of α, defined as values of α where the qualitative nature of the phase portrait for the system changes.

(c) Draw the phase portrait for values of α slightly larger and slightly smaller than each critical value of α.

Question 6. A mass m on a spring with spring constant k satisfies the differential equation
$$mu'' + ku = 0,$$
where $u(t)$ is the displacement at time t of the mass from its equilibrium position.

(a) Let $x_1 = u$ and $x_2 = u'$ and show that the resulting first-order system is
$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \mathbf{x}.$$

(b) Solve the system and draw a phase portrait.

(c) For one non-trivial trajectory (not the origin), draw the corresponding component functions as functions of time.

(d) Determine the relationship between the eigenvalues of the coefficient matrix and the frequency of the spring-mass system.

Question 7. For each system, solve for the general solution and determine a fundamental matrix for the system. Then also determine the fundamental matrix $\Phi(t)$, where $\Phi(0) = I_2$.

(a) $\mathbf{x}' = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} \mathbf{x}.$

(b) $\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ -1 & -1 \end{bmatrix} \mathbf{x}.$

(c) $\mathbf{x}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{x}.$

(d) $\mathbf{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix} \mathbf{x}.$

Question 8. Solve the IVP
$$\mathbf{x}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$
by using the fundamental matrix $\Phi(t)$, where $\Phi(0) = I_2$.