HOMEWORK PROBLEM SET 8: DUE APRIL 7, 2017

110.302 DIFFERENTIAL EQUATIONS
PROFESSOR RICHARD BROWN

Question 1. Let $x_1 = y$ and $x_2 = y'$ and convert the ODE

$$y'' + p(t)y' + q(t)y = 0$$

to a system of two first-order ODEs in x_1 and x_2. Then show that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions of your system, and if y_1 and y_2 form a fundamental set of solutions to the original ODE, then, up to a constant, the Wronskians are equal. That is, $W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = cW(y_1, y_2)$ for some non-zero constant $c \in \mathbb{R}$.

Question 2. Let $\mathbf{x}^{(1)}(t) = \begin{bmatrix} 2t \\ t^2 \end{bmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$ be two 2-vector functions. Do the following:

(a) Compute $W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ and determine all intervals where the functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent.

(b) Draw conclusions about the coefficients of the homogeneous system of equations satisfied by these two functions.

(c) Find a homogeneous system of first order linear ODEs where $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions.

Question 3. For each system, find a general solution, draw a direction field and plot enough trajectories to fully characterize the nature of the solutions to the system.

(a) $\mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -2 & 2 \end{bmatrix} \mathbf{x}.$

(b) $\mathbf{x}' = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix} \mathbf{x}.$

(c) $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{x}.$

(d) $\mathbf{x}' = \begin{bmatrix} -3 & -6 \\ 1 & 2 \end{bmatrix} \mathbf{x}.$ (Careful, here. This one has a 0-eigenvalue.)

Question 4. For each system, solve the IVP and describe the long term behavior (in both directions of the solution.

(a) $\mathbf{x}' = \begin{bmatrix} -5 & 1 \\ -3 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$

(b) $\mathbf{x}' = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}.$

Question 5. The following sets of eigenvalue/eigenvector pairs correspond to a system $\mathbf{x}' = A\mathbf{x}$. For each, (1) sketch a phase portrait for the system, (2) sketch the trajectory passing through the point $(1, 2)$ in the phase plane, and (3) Sketch separately the component functions $x_1(t)$ and $x_2(t)$, as functions of t on the same tx-plane.

(a) $r_1 = -1, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \quad r_2 = -2, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

(b) $r_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \quad r_2 = -2, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

(c) $r_1 = -1, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \quad r_2 = 2, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

(d) $r_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad r_2 = 2, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$

Question 6. Recall that the general solution to the ODE

$$ay'' + by' + cy = 0,$$

where a, b, c are constants and $a \neq 0$, depended on the solutions to the characteristic equation of the ODE. Transform this ODE into a 2-dimensional system of first-order ODEs by letting $x_1 = y$ and $x_2 = y'$ and show that the equation to determine the eigenvalues of the coefficient matrix is the same equation as the characteristic equation of the second-order ODE. The matrix equation to determine the eigenvalues is also called the characteristic equation of the matrix.

Question 7. For the 1-parameter system,

$$\mathbf{x}' = \begin{bmatrix} -1 & -1 \\ -\alpha & -1 \end{bmatrix} \mathbf{x},$$

do the following:

(a) Solve the system for $\alpha = \frac{1}{2}$ and classify the equilibrium solution at the origin. Sketch a phase portrait.

(b) Also solve the system and classify the equilibrium at the origin for $\alpha = 2$. again, sketch a phase portrait.

(c) Now solve the system for arbitrary α, obtaining the eigenvalues of the coefficient matrix as functions of α. Determine the value(s) of α where the classification of the equilibrium at the origin changes, and describe the transition from one classification to the other as the values of α pass through this value. Then attempt to sketch a phase portrait at this *bifurcation value* of α.