HOMEWORK PROBLEM SET 7: DUE MARCH 31, 2017

110.302 DIFFERENTIAL EQUATIONS
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Question 1. Solve the following:

- **(a)** $y''' - 3y'' + 3y' - y = 0$.
- **(b)** $\dfrac{d^4y}{dx^4} = 4\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2}$.
- **(c)** $\dddot{x} - \ddot{x} + \dot{x} - x = 0, \quad x(0) = 2, \quad \dot{x}(0) = -1, \quad \ddot{x}(0) = -2$.
- **(d)** $y''' - y'' - y' + y = 2e^{-t} + 3$.

Question 2. Determine the interval on which the solution to the IVP

$$(x-1)y^{(12)} + (x+1)y^{(4)} = (\tan x)y, \quad y(0) = 1, \quad y'(0) = \cdots = y^{(4)}(0) = 0,$$
$$y^{(5)}(0) = -6, \quad y^{(6)}(0) = \cdots = y^{(11)}(0) = \pi$$

is sure to exist and be unique.

Question 3. Verify that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the solution guess $y = v(t)y_1(t)$ leads to the following second order linear homogeneous ODE in v':

$$y_1 v''' + \left(3y_1' + p_1 y_1\right) v'' + \left(3y_1'' + 2p_1 y_1' + p_2 y_1\right) v' = 0.$$

Question 4. Transform the following into a system of first-order equations:

- **(a)** $t^2 y'' + ty' + \left(t^2 - \dfrac{1}{4}\right) y = 0$.
- **(b)** $u'' + p(t)u' + q(t)u = g(t), \quad u(0) = u_0, \quad u'(0) = u_0'$.

Question 5. For the system

$$\begin{aligned} \dot{x}_1 &= 3x_1 - 2x_2, \quad x_1(0) = 3 \\ \dot{x}_2 &= 2x_1 - 2x_2, \quad x_2(0) = \frac{1}{2}, \end{aligned}$$

do the following:

- **(a)** Transform the following system into a single equation of second order by solving the first equation for one of the variables and substituting this into the second equation, thereby creating a second order equation in one of the two variables.
- **(b)** Solve the second-order equation that you found in the previous part and then determine the solution for the other variable.
- **(c)** Find the particular solution and then graph it as a parameterized curve in the x_1x_2-plane, for $t \geq 0$.

Question 6. Consider the linear homogeneous system

$$\begin{aligned} x' &= p_{11}(t)x + p_{12}(t)y, \\ y' &= p_{21}(t)x + p_{22}(t)y. \end{aligned}$$

Show that if $x = x_1(t), y = y_1(t)$ and $x = x_2(t), y = y_2(t)$ are two solutions to the given system, then $x = c_1x_1(t) + c_2x_2(t), y = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants $c_1, c_2 \in \mathbb{R}$. This is again the Principle of Superposition.

Question 7. Show that the following vector functions solve the given ODE systems:

(a) $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t}.$

(b) $\mathbf{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x} = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t}.$

Question 8. For $A = \begin{bmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{bmatrix}$ and $B = \begin{bmatrix} i & 3 \\ 2 & -2i \end{bmatrix}$, find

(a) $A - 2B$.
(b) $3A + B$.
(c) BA.
(d) AB.

Question 9. Find all eigenvectors and eigenvalues of the matrices $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$.

Question 10. Solve the linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 1. \end{aligned}$$

Question 11. Either show that the following sets of vectors are linearly independent, or find a linear relation between them (the T means transpose):

(a) $\mathbf{x}^{(1)} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T, \quad \mathbf{x}^{(3)} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T.$

(b) $\mathbf{x}^{(1)} = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \quad \mathbf{x}^{(3)} = \begin{bmatrix} -1 & 2 & 0 \end{bmatrix}^T.$

Question 12. For

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} e^t \\ te^t \end{bmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix},$$

show that, for each choice of $t \in [0,1]$, the vectors $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linear dependent. Then show that as vector functions, $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent on $[0,1]$.