HOMEWORK PROBLEM SET 6: DUE MARCH 17, 2017

110.302 DIFFERENTIAL EQUATIONS
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Question 1. Solve the following using the Undetermined Coefficients Method:

(a) $y'' + 4y = t^2 + 3e^t$.

(b) $\ddot{x} + 2\dot{x} + 5x = 4e^{-t}\cos 2t, \quad x(0) = 1, \quad \dot{x}(0) = 0$.

(c) $y'' - y' - 2y = \cosh 2x$. Hint: Write the hyperbolic cosine function as a linear combination of exponentials.

Question 2. Solve the following using both the Variation of Parameters method and the Undetermined Coefficients Method:

(a) $4y'' + y = 4y' + 24e^{\frac{t}{2}}$.

(b) $\ddot{x} + 3\dot{x} + 2x = 3e^{-2t} + t, \quad x(0) = 0, \quad \dot{x}(0) = 0$. Notice here that, for a homogeneous ODE, the only solution that satisfies these initial conditions is the equilibrium $x(t) \equiv 0$. This is not true when the forcing function is not equivalently 0.

Question 3. Solve the following:

(a) $y'' + 2y' + y = \dfrac{e^{-x}}{x}$.

(b) $\ddot{r} + r = \csc t$.

Question 4. Show that the solutions to the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

can be written as $y(t) = u(t) + v(t)$, where u and v are solutions of the two initial value problems

$$\begin{aligned} L[u] &= 0, \quad u(t_0) = y_0, \quad u'(t_0) = y_0', \\ L[v] &= g(t), \quad v(t_0) = 0, \quad v'(t_0) = 0, \end{aligned}$$

respectively. Here, effectively, one can deal with the fact that the ODE is nonhomogeneous and an IVP as separate issues.

Question 5. One can use the Reduction of Order ideas for a nonhomogeneous ODE

$$y'' + p(t)y' + q(t)y = g(t),$$

when one solution y_1 of the corresponding homogeneous version of the ODE is known; again assume $y_2(t) = v(t)y_1(t)$ also solves the nonhomogeneous ODE and substitute it and its derivatives back into the ODE, creating a first-order ODE in v', as before. Do this for the ODE

$$ty'' - (1+t)y' + y = t^2e^{2t}, \quad t > 0; \quad y_1(t) = 1 + t.$$

Question 6. Consider the ODE

$$ay'' + by' + cy = g(t),$$

where a, b, c are positive constants. Do the following:

- **(a)** If $Y_1(t)$ and $Y_2(t)$ are both solutions, show that $Y_1(t) - Y_2(t) \longrightarrow 0$ as $t \to \infty$. Is this still true if $b = 0$?
- **(b)** If $g(t) = d$, a constant, show that every solution of the ODE approaches the constant $\frac{d}{c}$ as $t \to \infty$. What happens when $c = 0$? What happens if $b = c = 0$?

Question 7. A car supported by a MacPherson strut (shock absorber system) travels on a bumpy road at a constant velocity v. The equation modeling the motion of the car is

$$80\ddot{x} + 10000x = 2500\cos\left(\frac{\pi v t}{6}\right).$$

where x represents the vertical position of the cars axle relative to its equilibrium position, and the basic units of measurement are feet and feet per second (this is actually just an example of a forced, un-damped harmonic oscillator, if that is any help). The constant numbers above are related to the characteristics of the car and the strut. Note that the coefficient of time t (inside the cosine) in the forcing term on the right hand side is a frequency, which in this case is directly proportional to the velocity v of the car.

- **(a)** Find the general solution to this nonhomogeneous ODE. Note that your answer will have a term in it which is a function of v.
- **(b)** Determine the value of v for which the solution is undefined (you should present your final answer in miles per hour, as opposed to feet per second).
- **(c)** For a set of initial values $x(t) = \dot{x}(0) = 0$, graph the solutions for a few values of v near your answer in part **(b)** and not so near. Discuss the differences in these graphs and the importance of the special value of v in part **(b)**. (Hint: This special value of v induces what is called resonance in the car).