

HOMEWORK PROBLEM SET 11: DUE APRIL 28, 2017

110.302 DIFFERENTIAL EQUATIONS
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Question 1. For the following systems, verify that $\mathbf{x}^o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium and that the system is locally linear at the origin. Then classify the type and stability of the origin as best as one can by locally linearizing the system at the origin and classifying the linear equilibrium.

- (a) $\dot{x} = x - y^2, \quad \dot{y} = x - 2y + x^2.$
- (b) $\dot{x} = 2xy + y - x, \quad \dot{y} = x^2 - (y^2 + y + 4x).$

Question 2. Determine the periodic solutions, if any, of the system

$$\dot{x} = y + \frac{x}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2), \quad \dot{y} = -x + \frac{y}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2).$$

Question 3. For the system $\dot{x} = \epsilon x + y$ and $\dot{y} = -x + \epsilon y$, where ϵ is a parameter, classify the type and stability of the origin for different values of ϵ , and use this to verify that a center is not structurally stable; that neither the type nor the stability of a center persists under perturbations.

Question 4. For the competing species model $\dot{x} = x\left(\frac{3}{2} - \frac{1}{2}x - y\right)$ and $\dot{y} = y\left(2 - y - \frac{9}{8}x\right)$, draw a phase portrait and discuss the limiting behavior of the species populations $x(t)$ and $y(t)$ as $t \rightarrow \infty$ for various initial population sizes. Now do the same for the predator-prey model $\dot{x} = x\left(1 - \frac{y}{2}\right)$ and $\dot{y} = y\left(\frac{x}{2} - \frac{1}{4}\right)$.

Question 5. For the system $\dot{x} = y$ and $\dot{y} = x + 2x^3$, do the following:

- (a) Show that the origin is a saddle.
- (b) Sketch a phase portrait for the linearized system and show that all of the trajectories of the linear system that tend to the origin are on the line $y = -x$.
- (c) Sketch some trajectories of the nonlinear system integrating the corresponding first-order equation in $\frac{dy}{dx}$. In particular, sketch the trajectories that are asymptotic to the lines $y = x$ and $y = -x$.

Question 6. As in your last homework assignment, for each system, (1) find all critical points, (2) linearize the system around each critical point and solve the corresponding linear system, (3) use the linear system, as best as one can, to classify the type and stability of each nonlinear critical point. Then draw a small version of the linear phase portrait over the top of each nonlinear critical point in the the nonlinear phase portrait you drew for the last homework assignment.

- (a) $\frac{dx}{dt} = -xy + x, \quad \frac{dy}{dt} = y + 2xy.$
- (b) $\frac{dx}{dt} = 2x - x^2 - xy, \quad \frac{dy}{dt} = 3y - 2y^2 - 3xy.$

Question 7. Transform the system

$$\dot{x} = x - y - x(x^2 + y^2)$$

$$\dot{y} = x + y - y(x^2 + y^2)$$

into a system in polar coordinates to get $\dot{r} = r(1 - r^2)$ and $\dot{\theta} = 1$.

Question 8. Now for the system

$$\dot{x} = \alpha x - y - x(x^2 + y^2)$$

$$\dot{y} = x + \alpha y - y(x^2 + y^2),$$

where α is a parameter, do the following:

- (a) Show the origin is the only critical point for all values of $\alpha \in \mathbb{R}$.
- (b) Linearize the system at the origin and use it to determine the type and stability of the nonlinear equilibrium. How does this classification depend on α ?
- (c) Transform the system into polar coordinates, and explain how the phase portrait changes as the values of α change. Locate any bifurcation values of α and describe and draw a representative phase portrait on either side of each bifurcation value. (Note that the bifurcation you see here is called a *Poincaré-Andropov-Hopf bifurcation* or simply a *Hopf bifurcation*.)

Question 9. For each ODE system, find and classify all equilibria and cycles and draw a phase portrait:

(a) $\dot{r} = r(1 - r)(r - 2)(3 - r)(1 + r), \quad \dot{\theta} = -1.$

(b) $\dot{r} = r(1 - r)^2(r - 2), \quad \dot{\theta} = 4.$