HOMEWORK PROBLEM SET 10: DUE APRIL 21, 2017

110.302 DIFFERENTIAL EQUATIONS
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Question 1. For the following systems, find a general solution, draw a direction field and plot enough trajectories to fully characterize the nature of the solutions to the system.

(a) $\mathbf{x}' = \begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \mathbf{x}$.

(b) $\mathbf{x}' = \begin{bmatrix} -4 & 8 \\ -2 & 4 \end{bmatrix} \mathbf{x}$.

Question 2. Solve the IVP

$$\mathbf{x}' = \frac{1}{2} \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

and draw the solution in the phase plane. Then also graph each component of the solution as functions of t.

Question 3. Show that all solutions to

$$\mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}$$

approach the origin as $t \to \infty$ if and only if $a + d < 0$ and $ad - bc > 0$.

Question 4. For each of the following systems, do the following: (1) calculate the eigenvalues and eigenvectors; (2) classify the equilibrium at the origin, both in its type and its stability; and (3) Sketch a phase portrait.

(a) $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$.

(b) $\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}$.

(c) $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$.

(d) $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{x}$.

Question 5. The linear, second-order, homogeneous, autonomous, ODE with constant coefficients

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0,$$

where $m, c, k > 0$ are all constants, models a spring-mass system with damping. Write this ODE as a first-order system, where $x = u$, and $y = u'$. Show that the origin ($x = 0$ and $y = 0$) is a critical point, and analyze the nature and stability of this critical point as a function of the parameters m, c, and k.

Question 6. (What Figure 9.1.9 on page 507 of the text means.) Given the ODE system

$$\mathbf{x}' = A\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x},$$

let $p = \operatorname{tr} A = a + d$, and let $q = \det A = ad - bc$, and finally let $\Delta = p^2 - 4q$. Show that the critical point at the origin is a:

(a) Node if $q > 0$ and $\Delta \geq 0$;
(b) Saddle if $q < 0$;
(c) Spiral Node if $p \neq 0$ and $\Delta < 0$;
(d) Center if $p = 0$ and $q > 0$.
(e) Also show that, if the two solutions to the characteristic equation of A are r_1 and r_2, that it is always the case that $p = r_1 + r_2$ and $q = r_1 r_2$.
(f) Show that the critical point at the origin is asymptotically stable if $q > 0$ and $p < 0$.
(g) Show that the critical point at the origin is stable if $q > 0$ and $p = 0$.
(h) Show that the critical point at the origin is unstable if $q < 0$ or $p > 0$.

Question 7. For each system, (1) find all critical points, (2) use a computer or hand draw a direction field and sketch a few trajectories near the critical points, and (3) determine as best as one can the type and stability of each critical point.

(a) $\frac{dx}{dt} = -xy + x, \quad \frac{dy}{dt} = y + 2xy.$
(b) $\frac{dx}{dt} = 2x - x^2 - xy, \quad \frac{dy}{dt} = 3y - 2y^2 - 3xy.$

Question 8. For each system, (1) find a function $H(x, y) = c$ satisfied by the trajectories, and (2) plot several level curves of H, indicting the direction of travel for increasing t.

(a) $\frac{dx}{dt} = 2y, \quad \frac{dy}{dt} = 8x.$
(b) $\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x$ (this is the undamped pendulum.)