

Here is a much better way to characterize this:

Proposition. A system $\vec{x}' = \vec{F}(\vec{x}) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$ is almost linear at an isolated critical pt $\vec{x} = \vec{x}^0$ if both F, G have continuous partials up to order 2, and $D\vec{F}(\vec{x}) = \begin{bmatrix} \frac{\partial F}{\partial x} |_{\vec{x}^0} & \frac{\partial F}{\partial y} |_{\vec{x}^0} \\ \frac{\partial G}{\partial x} |_{\vec{x}^0} & \frac{\partial G}{\partial y} |_{\vec{x}^0} \end{bmatrix}$ is non-degenerate.

Notes ① Here, $D\vec{F}(\vec{x})$ is the derivative of $\vec{F}$ as a vector-valued function on a domain in $\mathbb{R}^2$, called the Jacobian of $\vec{F}$ at $\vec{x}^0$. It becomes the matrix of the associated linearized system.

② Indeed, change variables to 'move' $\vec{x}^0$ to the origin: let $\vec{u} = \vec{x} - \vec{x}^0 = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$, $\vec{u}' = \vec{x}'$

So let $\vec{x}' = \vec{F}(\vec{x})$ have a critical pt at $\vec{x}^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

Then for $u_1 = x - x_0$, $u_2 = y - y_0$, we have $u_1' = x'$, $u_2' = y'$, and.

Then

$$\begin{aligned}
 \vec{u}' &= \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} F(x-x_0, y-y_0) \\ G(x-x_0, y-y_0) \end{bmatrix} \\
 &= \underbrace{\begin{bmatrix} F(x_0, y_0) + F_x(x_0, y_0)(x-x_0) + F_y(x_0, y_0)(y-y_0) \\ G(x_0, y_0) + G_x(x_0, y_0)(x-x_0) + G_y(x_0, y_0)(y-y_0) \end{bmatrix}}_{1^{\text{st}} \text{ Taylor Polynomial of } F \text{ at } \vec{x}^0} + \underbrace{\begin{bmatrix} \eta_1(x, y) \\ \eta_2(x, y) \end{bmatrix}}_{\text{Higher order stuff.}}
 \end{aligned}$$

$$= \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \begin{bmatrix} \eta_1(x, y) \\ \eta_2(x, y) \end{bmatrix}$$

$$\vec{u}' = A \vec{u} + \vec{g}(\vec{u})$$

and $\vec{g}(\vec{u})$ will satisfy the smallness property since 2nd derivatives also exist.

ex. Back to example $\dot{x} = 4 - 2y$
 $\dot{y} = 12 - 3x^2$. Here P_0

is fixed at $(2, 2)$. Transform $u = x - 2$
 $v = y - 2$.

Here $\dot{u} = \dot{x}$, $\dot{v} = \dot{y}$, and $\dot{u} = -4 - 2(v+2) = -2v$
 $\dot{v} = 12 - 3(u+2)^2 = -12u - 3u^2$

$$\text{Then } \vec{u}' = A \vec{u} + \vec{g}(\vec{u}) = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix} \vec{u} + \begin{bmatrix} 0 \\ -3u^2 \end{bmatrix}$$

$$\text{So } A = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}, \quad \vec{g}(\vec{u}) = \begin{bmatrix} 0 \\ -3u^2 \end{bmatrix}.$$

Since $\|\vec{g}(\vec{u})\| = \sqrt{0^2 + (-3u^2)^2} = 3u^2$, and

$$\|\vec{u}\| = \sqrt{u^2 + v^2} = r, \text{ where } \begin{matrix} u = r \cos \theta \\ v = r \sin \theta \end{matrix}, \text{ we have}$$

$$\lim_{\vec{u} \rightarrow \vec{0}} \frac{\|\vec{g}(\vec{u})\|}{\|\vec{u}\|} = \lim_{r \rightarrow 0} \frac{3r^2 \cos^2 \theta}{r} = \lim_{r \rightarrow 0} 3r \cos^2 \theta = 0$$

Hence system (1) almost linear at $(2, 2)$.

Thm Let r_1, r_2 be the eigenvalues of the linear system $\vec{x}' = A\vec{x}$ corresponding to the almost linear $\vec{x}' = A\vec{x} + \vec{g}(\vec{x})$. Then

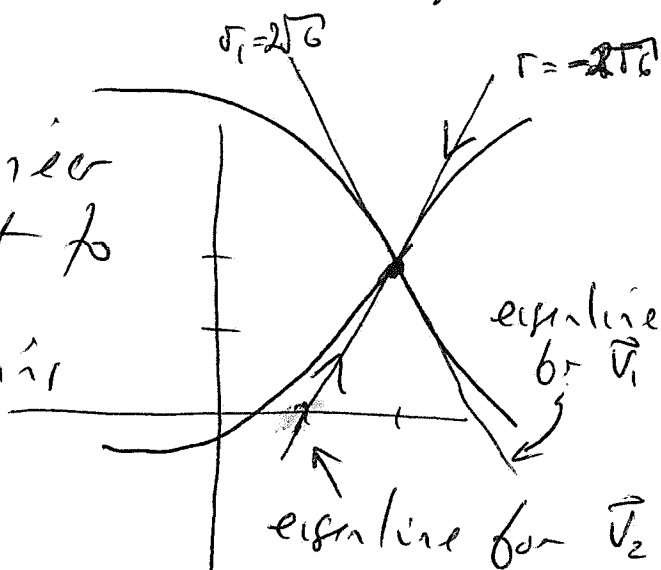
- ① If $r_1 \neq r_2$ not purely imaginary, the stability and type of the fixed pts are the same.
- ② If $r_1 = r_2$, the fixed pt of the nonlinear system has the same stability as that of the linear system but not necessarily the type.
- ③ If $r = \pm i\mu$, then neither the stability nor the type of the nonlinear critical pt can be determined by the linear system.

In the example, $A = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}$ has eigenvalues $\lambda = \pm 2\sqrt{6}$.

Hence the linear system has a saddle at the origin. By the Thm, then, so does the original nonlinear system at $(2, 2)$.

Furthermore, for $\lambda_1 = 2\sqrt{6}$, $\lambda_2 = -2\sqrt{6}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ \sqrt{6} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -\sqrt{6} \end{bmatrix}$

To linear approximation, even the phase portraits are the same. The eigenlines of the linear system are tangent to the level set passing through $(2, 2)$.



exercise: Do this for the fixed pt at $(-2, 2)$. The linear system has a center at the origin. But by the Thm we cannot tell if the original is a center or not.

Note: As long as F, G are "nice" (have cont. derivatives up to and including order 2) at a crit. pt, the system is almost linear.

One calculates $A = \begin{bmatrix} F_x|_{\vec{x}^0} & F_y|_{\vec{x}^0} \\ G_x|_{\vec{x}^0} & G_y|_{\vec{x}^0} \end{bmatrix}$ and there is no need to worry about the $\vec{g}(\vec{x})$.

ex. Determine the stability of $\vec{x}^0 = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$ of $\begin{cases} \dot{x} = (1+x)\sin y \\ \dot{y} = 1-x-\cos y \end{cases}$

Soln: Both $F(x,y) = (1+x)\sin y$
 $G(x,y) = 1-x-\cos y$ are C^∞ , so system is almost linear everywhere.

Here $\vec{F}(\vec{x}^0) = F(2,\pi) = 0 = G(2,\pi)$, and

$$A = \begin{bmatrix} F_x(2,\pi) & F_y(2,\pi) \\ G_x(2,\pi) & G_y(2,\pi) \end{bmatrix} = \begin{bmatrix} \sin \pi & (1+2)\cos \pi \\ -1 & \sin \pi \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues of A satisfy $\Gamma^2 - 0\Gamma + 3 = 0$ or $\Gamma = \pm\sqrt{3}$

Hence $\vec{x}^0 = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$ is a saddle and unstable.

X

ex. Determine the stability of the fixed pt at $(0,0)$ of the system $\dot{x}=y$, $\dot{y}=-\delta y - \omega^2 \sin x$ for δ, ω^2 positive constants (for different values of each).

Notes: This system is almost linear everywhere since $F(x,y)=y$, $L(x,y)=-\delta y - \omega^2 \sin x \in C^\infty$.

Soln: At $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x & -\delta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\delta \end{bmatrix}$.

with eigenvalues satisfying $r^2 + \delta r + \omega^2 = 0$

or $r = \frac{-\delta \pm \sqrt{\delta^2 - 4\omega^2}}{2}$.

For different values of δ, ω^2 , what can we say about the equilibrium at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

The cases here seem to imply that the magnitude and/or sign of $\delta^2 - 4\omega^2$ will determine Γ and hence the type and possibility of the origin.

Cases

(I) Suppose $0 < \gamma < 2|w|$. Then eigenvalues are complex with real part $-\frac{\gamma}{2} < 0$.

Then the corresponding linear system has a spiral sink at (0) . By the thm, then, so does the nonlinear system.

(II) $\gamma = 2|w|$
 (III) $\gamma > 2|w|$ } Try to classify and/or understand these 2 cases.