

Back to $\vec{x}' = A_{2 \times 2} \vec{x}$ in the case where the 2 eigenvalues of A , r_1, r_2 are real and distinct: $r_1 \neq r_2$.

Some Notes

① This works fine when 0 is an eigenvalue:

ex. $\vec{x}' = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \vec{x}$. Here char. eqn of $\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$

is $r^2 - 2r = 0$, w/ solns $r_1 = 0, r_2 = 2$.

For $r_1 = 0$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a choice of eigenvector.

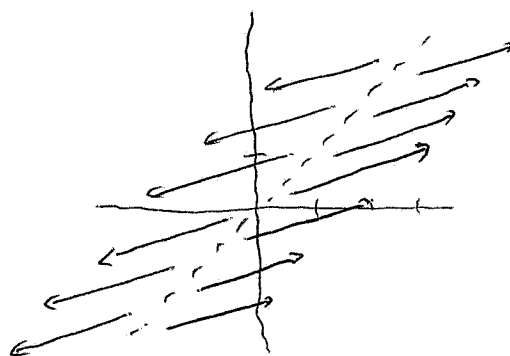
For $r_2 = 2$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is one choice

$$\begin{aligned} \text{Then } \vec{x}(t) &= c_1 \vec{v}_1 e^{r_1 t} + c_2 \vec{v}_2 e^{r_2 t} \\ &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t} \text{ is gen. soln.} \end{aligned}$$

Q: What does the phase portrait look like here?

Special Note: Here let $A = 0$

but there are still 2 independent eigenvectors!



- ② This formulation of the general solution still works even for repeated eigenvalues if there are enough eigenvectors.

ex. $\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$. Here characteristic eqn

$$15 \quad r^2 - 2r + 1 = 0 = (r-1)^2. \text{ Hence}$$

$$r_1 = r_2 = 1.$$

The eigenvector eqn $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

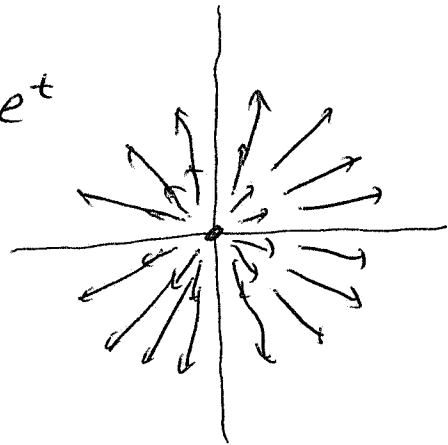
$\Rightarrow \begin{matrix} x_1 = x_1 \\ x_2 = x_2 \end{matrix}$ can be solved using 2 lin ind. vectors.

One choice is $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Hence ~~there~~

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$$

is the general solution.



ex. (pg. 403) $\vec{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$

Here $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\det(A - \Gamma I) = \begin{vmatrix} -\Gamma & 1 & 1 \\ 1 & -\Gamma & 1 \\ 1 & 1 & -\Gamma \end{vmatrix}$

$$= -\Gamma \begin{vmatrix} -\Gamma & 1 \\ 1 & -\Gamma \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -\Gamma \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ -\Gamma & 1 \end{vmatrix}$$

$$= -\Gamma(\Gamma^2 - 1) - 1(-\Gamma - 1) + 1(1 + \Gamma) = 0$$

$$= -\Gamma^3 + \Gamma + \Gamma + 1 + 1 + \Gamma = \Gamma^3 - 3\Gamma + 2 = 0$$

By long division

$$\Gamma^3 - 3\Gamma + 2 = (\Gamma + 1)(\Gamma^2 - \Gamma - 2) = (\Gamma + 1)^2(\Gamma - 2)$$

Hence eigenvalues are $\Gamma_1 = 2, \Gamma_2 = \Gamma_3 = -1$

In the book, they solve $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

So general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}$$

exercise: Imagine the 3-dim phase space...

What happens near the equilibrium at the origin?

- ③ This formulation for finding solutions does not work when there are not enough ~~eigenvectors~~ independent eigenvectors for repeated eigenvalues.

ex. $\vec{x}' = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{x}$. Here, eigenvalues $\lambda_1 = \lambda_2 = -2$.

But the eigenvector eqn $A\vec{v} = -2\vec{v}$, or

$$-2x_1 + x_2 = -2x_1$$

$$-2x_2 = -2x_2$$

is only solved by $x_2 = 0$, $x_2 \neq 0$. An only independent choice would be $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Here, we will need to come up with another method to find another independent soln.

Properties of phase portraits

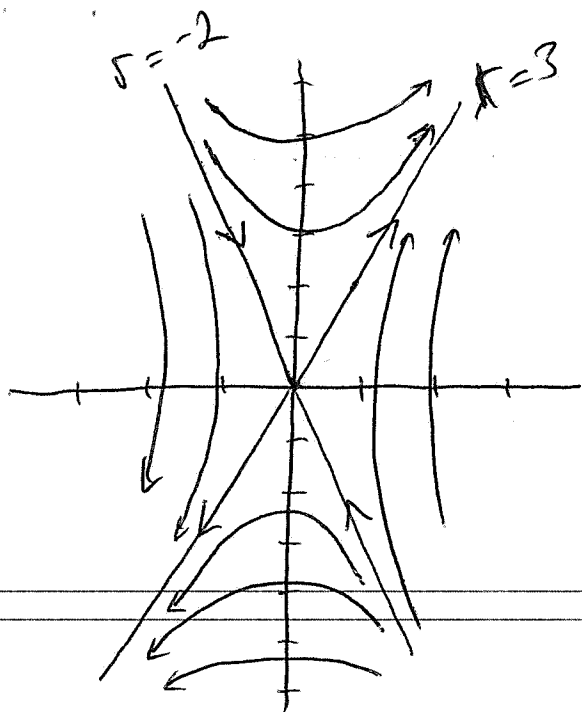
Let $\vec{x}' = A\vec{x}$. We have

① For any $A_{2 \times 2}$, the origin is an equilibrium solution ($\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is always a solution to $A\vec{x} = \vec{0}$).

② If $\det A \neq 0$, then $\vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the ONLY equilibrium solution (stationary pt, or fixed pt), since $A\vec{x} = \vec{0}$ has $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as the only solution.

③ The eigenvectors of A correspond to lines through the origin where the solutions exhibit straight line motion

Note: These straight lines contain many distinct solutions.



④ The sign of each eigenvalue determines direction along lines (toward origin if < 0 outward if > 0).

phase portrait for
 $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$.

w/ solns

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$$

⑤ If eigenvalues are real and distinct then these are the only straight lines (why?)

⑥ Signs of eigenvalues determine "stability" of equilibrium at origin

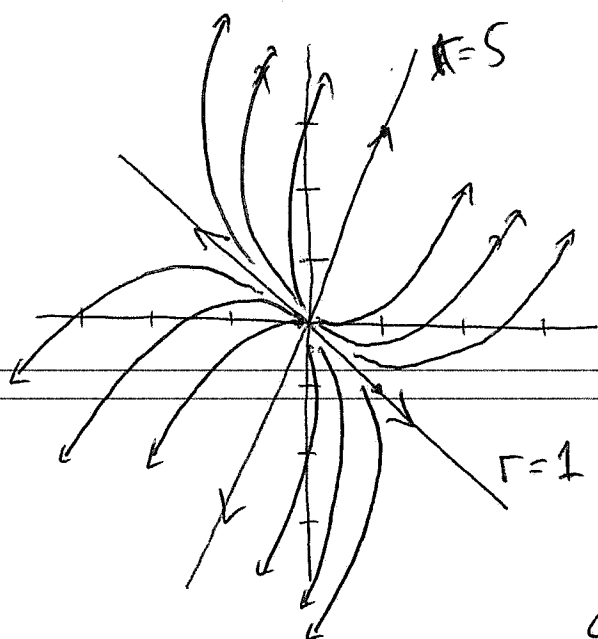
(How solutions behave "near" the origin. Do they stay nearby, converge to, or diverge from....)

⑦ Above, origin is called a "saddle pt".

Would you consider it stable?

ex. $\vec{x}' = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \vec{x}$. Here $r_1 = 5, r_2 = 1$,

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$



The general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

All solutions tend to origin as $t \rightarrow -\infty$ and are unbounded as $t \rightarrow \infty$

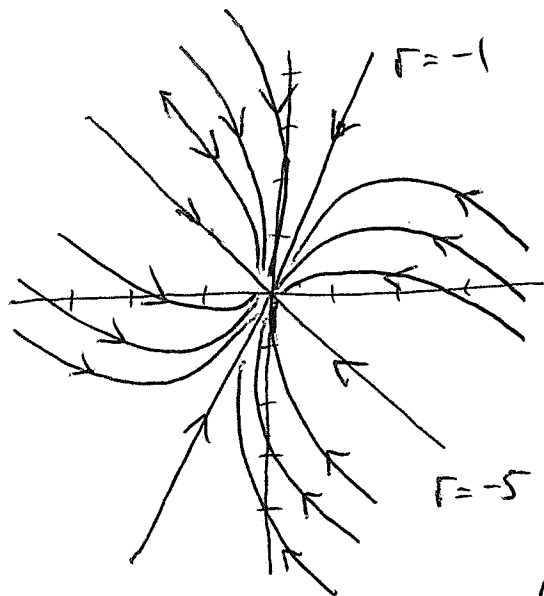
Origin is called a source and is unstable.

Q: Why is motion curved like above? How to determine?

ex. $\vec{x}' = \begin{bmatrix} -4 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}$

Here $r_1 = -5, r_2 = -1$, with

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



General solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

Phase portrait is similar to above but different: how?

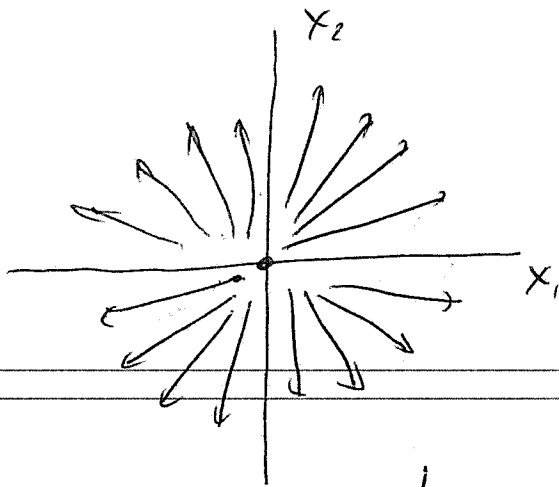
Here origin is a "sink" and asymptotically stable.

ex. $\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$

Here, as we have seen

$$r_1 = r_2 = 1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{and } \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \\ = (c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}) e^t$$



This is another source at the origin here called a star node.

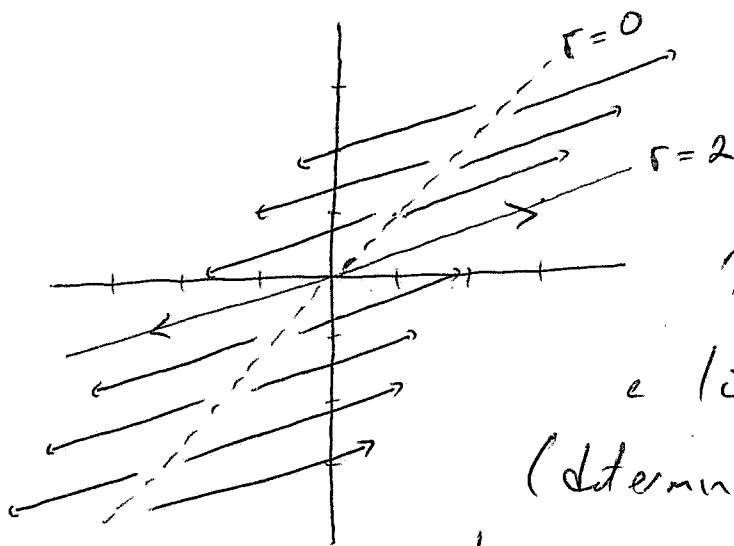
ex. $\vec{x}' = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \vec{x}$

Here $r_1 = 0, r_2 = 2,$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

General soln is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$$



This one is special: There is a line of equilibrium solution (determined is 0). off the dotted line, motion is straight along lines parallel to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and moving out from dotted line.

Q: What is the stability?