

~~Math~~ 110.302 Lecture 10: ~~Math~~ I

I will relegate the discussion of Section 2.8 to a worksheet posted. The theory can be deep (though very interesting).

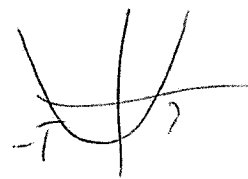
But the main takeaway is its usefulness in

- ① seeing where e.g. 1st order ODE is "nice"
 - ② Understanding the intricacies of the theory even at this early stage.
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Section 3.1 The general form for a 2nd order ODE is

$$(*) \quad y'' = f(t, y, y')$$

for some function f .



$$(x-5)(x+3)$$

$$x=5 \quad x=-3$$

Def A 2nd order ODE is called linear II
if it can be written

$$P(t)y'' + Q(t)y' + R(t)y = Q(t)$$

or more familiarly

$$y'' + p(t)y' + q(t)y = g(t)$$

where $p(t) = \frac{Q(t)}{P(t)}$, $q(t) = \frac{R(t)}{P(t)}$, $g(t) = \frac{Q(t)}{P(t)}$

Notes ① Here in (*), this means

$$A(t, y, y') = g(t) - p(t)y' - q(t)y$$

and A is linear in y and y' .

② If not linear, then non-linear

Def If $g(t) \equiv 0$, then ODE is called
homogeneous.

Def An IVP with (*) contains 2 pieces
of initial data, usually

$$y(t_0) = y_0, \quad y'(t_0) = y'_0 \quad \text{why?}$$

In general, it is hard or impossible to solve
a 2nd order ODE

Even linear is very difficult in general!

One type that is solvable: Constant coefficients

Let $(**)$ have $P(t) \equiv a$, $Q(t) \equiv b$, $R(t) \equiv c$,
and suppose $Q(t) \equiv 0$ (homogeneous).

Then ODE is $ay'' + by' + cy = 0$ (A)

Q: First think, what kinds of functions would
possibly be solutions to this kind of ODE?

- Polynomials? Power Functions?
- Trig functions?
- Exponentials?
- Logarithms?

ex Suppose $a=1, b=0, c=-1$. Then (**) is $y'' - y = 0$, or $y'' = y$.

Solutions! Here $y(t) = e^t$ and $y(t) = e^{-t}$ both solve $y'' - y = 0$.

How about $e^t + e^{-t}$? $2e^t - 3e^{-t}$?

Here $y(t) = c_1 e^t + c_2 e^{-t}$ is a solution for any choice of $c_1, c_2 \in \mathbb{R}$.

What if IVP was $y'' - y = 0, y(0) = 3, y'(0) = 4$?

Then $y(0) = 3 = c_1 e^0 + c_2 e^{-0} = c_1 + c_2$

and $y'(0) = 4 = c_1 e^0 - c_2 e^{-0} = c_1 - c_2$

$$\left. \begin{array}{l} 3 = c_1 + c_2 \\ 4 = c_1 - c_2 \end{array} \right\} \begin{array}{l} c_1 = \frac{7}{2} \\ c_2 = -\frac{1}{2} \end{array}$$

And the particular solution to IVP is

$$y(t) = \frac{7}{2} e^t - \frac{1}{2} e^{-t}.$$

ex 2 $2y'' + 8y' - 10y = 0$

Here $y(t) = e^t$ and $y(t) = e^{-5t}$ both work! Check this....

AND so does $y(t) = c_1 e^t + c_2 e^{-5t}$.

Is there a pattern?

For $ay'' + by' + cy = 0$, assume the solution is exponential (is this a good idea?) and looks like $y(t) = e^{\gamma t}$ where γ is an unknown parameter.

Then substituting $y(t)$ and its derivatives into the ODE, we get

$$a\gamma^2 e^{\gamma t} + b\gamma e^{\gamma t} + ce^{\gamma t} = 0$$

(*) or $a\gamma^2 + b\gamma + c = 0$. Any valid values for γ must satisfy $\gamma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Recognize this?

For a 2nd order homogeneous ODE w/ constant coefficients, (*) is called the characteristic equation.

Important Fact

When (*) has 2 solutions r_1, r_2 which are real and distinct ($r_1 \neq r_2$), then the general solution to $ay'' + by' + cy = 0$ is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

(we will show this later).

ex1. $y'' - y = 0$, with $a=c=1, b=0$.

characteristic eqn. is $r^2 - 1 = 0$, solved by $r = -1, 1$. Gen. soln is $y(t) = c_1 e^t + c_2 e^{-t}$

ex2 Characteristic eqn of $2y'' + 8y' - 10y = 0$ is $2r^2 + 8r - 10 = 0 = (2r-2)(r+5)$. Hae gen soln is $y(t) = c_1 e^t + c_2 e^{-5t}$.