Lecture 1:

Start with $y = f(x)$, some unknown functional relation between 2 variables, where

x - independent variable
y - dependent variable

Such an equation is (or sets of them are) called a <u>mathematical model</u> when the variables represent measurable quantities in some application (usually set up to study some unknown entity y based on its relationship to something controllable x).

If $y = f(x)$ is known, then we can simply study its properties (using calculus).

Often, though, we do not know $y = f(x)$, but we do have information about some properties, like derivatives, for example.

examples

(I) $\frac{dy}{dx} = ky, \quad k \in \mathbb{R}.$

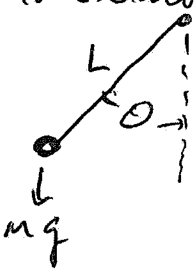
(II) $F = ma$ (Newton's 2nd Law of motion).

(III) $f'(x) = x - e^{x/2}$ (restatement of: Find $\int (x - e^{x/2})\,dx$).

or $y' = g(x)$, $y(x)$ is unknown soln: $g(x) = \int g(x)$

(IV) $$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

The Pendulum.



The above are mathematical models where the actual function is only known implicitly....

Def An ordinary differential equation (ODE) is an equation involving an unknown function between 2 entities and some of its derivatives.

Note: "Ordinary" means that the unknown function is a function of one independent variable.

ex. The Heat Equation (in 3-space)

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} + \frac{\partial u^2}{\partial z^2}\right)$$

is a partial diff. eqn since u is a fnc of more than 1-indep. variable.

Def. The order of an ODE is the same as the order of the highest derivative that appears in the equation:

ex (I) and (III) are first order ODEs.
(II) and (IV) are 2^{nd} order ODEs (do you see this?)

Def The general form of an nth order ODE is

$$F(x, y, y', \dots, y^{(n)}) = 0 \qquad (*)$$

where x - indep var (time?)
y - dep var (unknown fnc $y = f(x)$)
$y^{(i)}$ is the ith derivative of $y = f(x)$
F is some expression in $x, y, y', \dots, y^{(n)}$

Note: Sometimes, we can solve for the highest derivative:

$$(**) \qquad y^{(n)} = h(x, y, y', \ldots, y^{(n-1)}),$$

But not always:

ex. $y^{(5)} + \sin y^{(5)} = y^{(3)}$ cannot be written like $(**)$. But for $(*)$, $F = y^{(5)} + \sin y^{(5)} - y^{(3)}$.

Def. A function $f(\underbrace{x_1, \ldots, x_n}_{\vec{x}}, y_1, \ldots, y_m)$, or eqn $f(\vec{x}, \vec{y}) = 0$

is linear in the variables $y_1, \ldots, y_m$ if

$$f(x_1, \ldots, x_n, y_1, \ldots, y_m) = a_0(\vec{x}) + \sum_{i=1}^{m} a_i(\vec{x}) y_i$$

where the $a_i(x)$, $i = 0, \ldots, m$ are arbitrary.

Notes ① For an ODE to be linear, it must be linear in $y, y', \ldots, y^{(n)}$, and can be written as

$$a_n(\vec{x}) y^{(n)} + \cdots + a_1(\vec{x}) y' + a_0(\vec{x}) y = g(x).$$

examples

(A) $(\sin x)y' + (\ln x)y = \tan e^x$ is a linear 1^{st} order ODE.

(B) $y'' + xy' + \sin y = 0$ is NOT linear

(C) $y''y + y' = 0$ is Not linear.

Suppose $y' = f(t,y)$ is a 1^{st} order linear ODE (written like (**)).

Then there exist functions $p(t)$, $q(t)$ so that $f(t,y) = -p(t)y + q(t)$, and the ODE can be written as

$$y' + p(t)y = q(t)$$

This form will be very important to understanding how to study this type of ODE.

ex. Given $(\sin x)y' + (\ln x)y = \tan e^x$, identify $p(t)$.

Solution: Divide by $\sin x$ to get,

$$y' + \left(\frac{\ln x}{\sin x}\right)y = \frac{\tan e^x}{\sin x}. \quad p(t) = \frac{\ln x}{\sin x}.$$

Def. A <u>solution</u> to $(*)$ on $I=(a,b)\subset\mathbb{R}$ is any function $y=f(x)$ that satisfies the equation $(*)$.

<u>ex.</u> $\frac{dp}{dt}=\frac{p}{2}-450$ is solved by $p(t)=900+ce^{t/2}$ $\forall c\in\mathbb{R}$.

- How do we know? Try it.
- How did we find it? <u>keep listening</u>.
- What to make of the parameter c?

<u>ex.</u> Sometimes a solution is only known <u>implicitly</u>:

Show $x^2+y^2-5=0$ solves $\frac{dy}{dx}=-\frac{x}{y}$.

Here only locally can we solve for $y(x)$.

<u>ex.</u> Solve $x''(t)+\frac{k}{m}x(t)=0$.

<u>ex</u> Solve $y'(x)=x-e^{x/2}$

110.302 Lecture 2: Wed, ~~[illegible]~~

Compare the following:

(a) $\frac{dy}{dx} = x - e^{x/2}$ (b) $\frac{dp}{dt} = \frac{p}{2} - 450$

(1) Both are ~~[illegible]~~ First order ODEs.

(2) Both are linear (can you ~~[illegible]~~ see this?)

(3) (a) is of the form $y' = f(x)$ ← independent variable on RHS

and is simply a calculus problem

(think: find $\int f(x)\,dx$) ~~[illegible]~~.

Here the RHS is only a fnc of the independent variable.

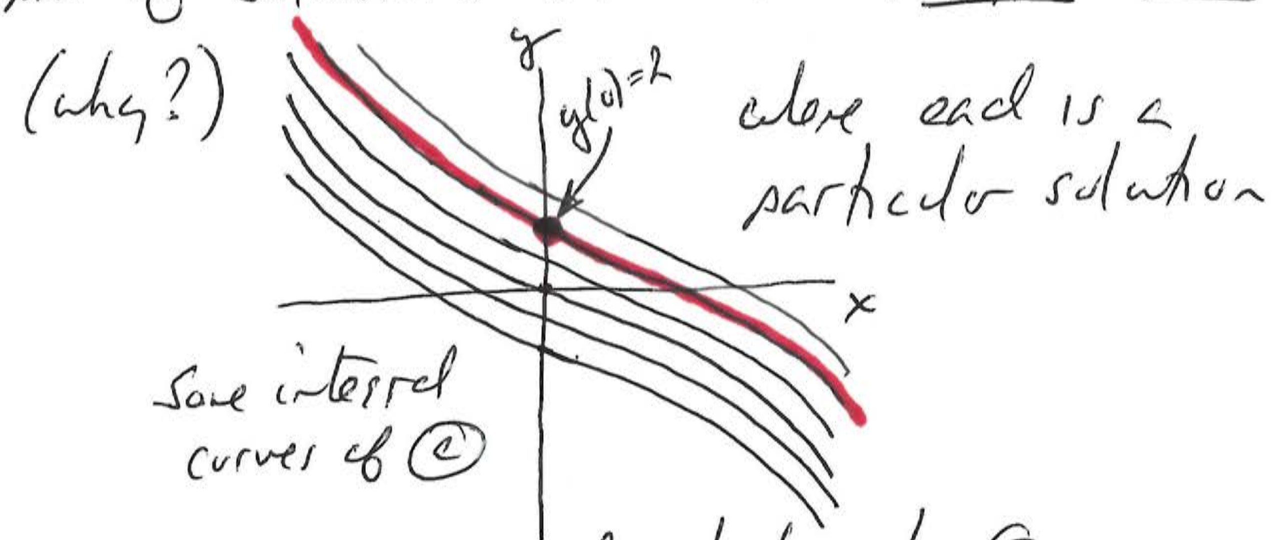
integrate the equation. What does that mean?

- Integrate both sides with respect to x to get

$$y(x) = \frac{x^2}{2} - 2e^{x/2} + C$$

- This formula is called a <u>general solution</u> to (a) as it is an expression that specifies <u>all</u> possible solutions.

- A particular solution to ⓐ involves choosing a value for the constant c.
- Graphs of solutions are called integral curves (why?)



- We also call the general solution to ⓐ a 1-parameter family of solutions.

With the addition of a single pt in the xy-plane (ex. $y(0)=2$), called an initial value, we can "choose" a particular solution from the family.

With this point, there is now only 1 solution to the problem:

$$y(x) = \frac{x^2}{2} - 2e^{x/2} + C$$

$$y(0) = \frac{0^2}{2} - 2e^{0/2} + C = 2$$

$$= 0 - 2 + C = 2 \Rightarrow C = 4$$

Hence the solution to the $\underbrace{\text{Initial Value Problem}}_{\text{(IVP)}}$ (an ODE with initial values)

$$\underbrace{y' = x - \frac{e^{x}}{2},\ y(0) = 2}_{\text{IVP}} \quad \text{is} \quad y(x) = \frac{x^2}{2} - 2e^{x/2} + 4.$$

See the red curve above.

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(4) (B) is not of the form $y' = f(x)$.

Rather, it is of the form $y' = f(y)$ ← dependent variable on RHS.

(in this case $p' = f(p)$)

Note: This is harder to solve but easier to study!

Def if in an ODE the independent variable is NOT explicitly present, the ODE is called *autonomous*.

Let's solve ⑥: There are many ways. We choose something like what we will eventually call "separation of variables".

First, recall from Calculus I,

Ⓘ For any diff. $p(t)$, the functions $p(t)$ and $p(t)-900$ have the same derivative: $p'(t)$.

Ⓘⓘ $\frac{d}{dt}\left[\ln|f(t)|\right] = \frac{f'(t)}{f(t)}$ for $f(t)$ differentiable and $f(t)\neq 0$.
(chain Rule)

Hence we can rewrite $p' = \frac{p}{2} - 450 = \frac{p-900}{2}$.

as $\frac{p'}{p-900} = \frac{1}{2}$. Why is this useful?

it is useful since the LHS of $\frac{p'}{p-900} = \frac{1}{2}$
looks like the derivative

$$\frac{d}{dt}\left[\ln|p(t)-900|\right] = \frac{1}{2}.$$

Integrate both sides as functions of t to get

$$\ln|p(t)-900| = \frac{t}{2} + c$$

and exponentiate to get ~~[illegible]~~

$$|p(t)-900| = e^{t/2+c} = e^{t/2}e^{c}$$ make sense?

If we can "solve" this for p(t) we are done
since we would have an expression for
the p(t) that solves (6).

Q: Given $|p(t)-900| = e^{t/2}e^{c}$, can $p(t)=900$?
Why or why not?

The ODE ① has 3 types of solutions:

① $p(t) > 900$ always $\Rightarrow p(t) - 900 = Ke^{t/2}$

or $p(t) = 900 + Ke^{t/2}$, where $K = e^{c} > 0$.

② $p(t) < 900$ always $\Rightarrow -(p(t) - 900) = Ke^{t/2}$

or $p(t) = 900 + \underbrace{(-K)}_{\text{negative constant}} e^{t/2}$, where $K = e^{c} > 0$.

③ $p(t) = 900$ for all $t \in \mathbb{R}$. It works in the ODE and can also be written

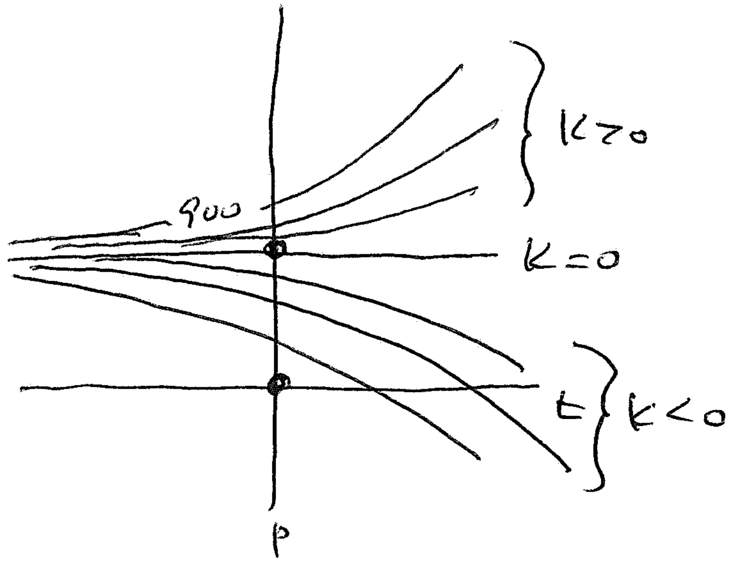
$$p(t) = 900 + Ke^{t/2}, \quad K = 0.$$

This last solution is called a <u>singular solution</u> which sometimes become hidden due to the method employed to find the other solutions.

Note: Our method included a divide by $p - 900$ term which implied we discounted that possibility. We need to account for it.

Conclusion $p(t) = 900 + Ke^{t/2}$, $K \in \mathbb{R}$

is the general solution to (b)



Here the singular solution $p(t) = 900 \ \forall t$ is called an equilibrium solution (or a steady-state soln).

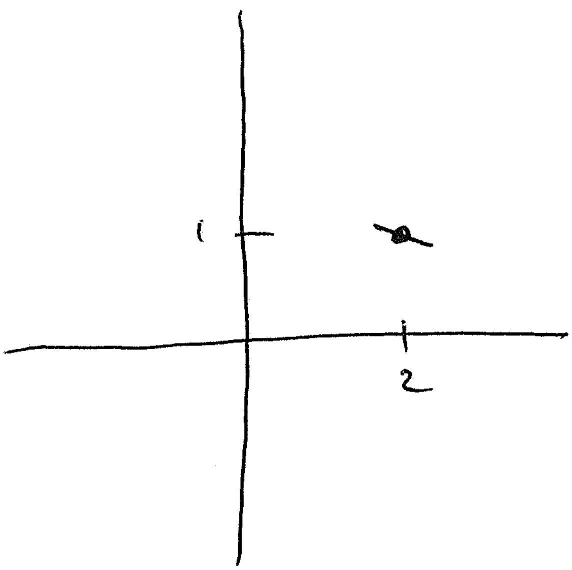
Equilibrium solutions will be very important to us in the future.

Notice in both (a) and (b) above

$$\frac{dy}{dx} = \text{something involving } x, y \qquad \frac{dp}{dt} = \text{something involving } t, p$$

This is useful since solutions "live" in the xy-plane or the tp-plane, any solution curve will have its tangent line at (x,y) with slope given by simply evaluating the RHS at that point.

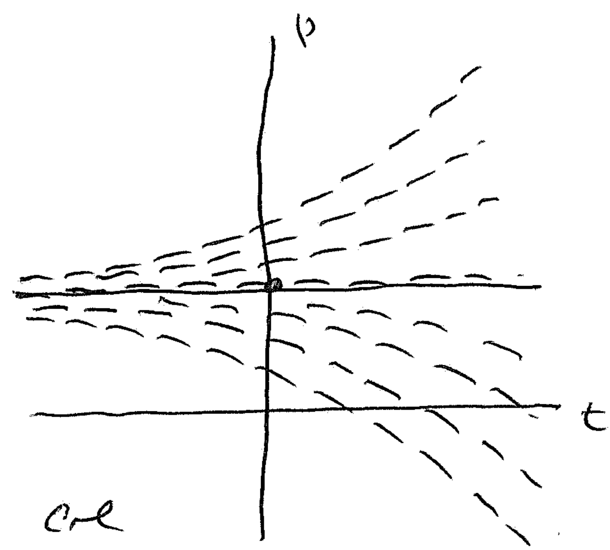
ex. For $y' = x - e^{y/2}$, choose $(x_0, y_0) = (2, 1)$.



Then $\left.\dfrac{dy}{dx}\right|_{\substack{x=2\\y=1}} = 2 - e^{2/2} \cong -.718$

The solution curve passing through $(2,1)$ will have slope $\sim -.718$ there.

Without solving the ODE, we can take a grid of pts in the xy-plane and evaluate these little slope lines. The result is called a slope field:



like blades of grass in a stream with a strong current.

These line segments are tangent to all solution curves.

We will use these as a method of study over time.